ON TIES IN TRIPLE COMPARISONS

By

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THIS DISSERTATION IS DEDICATED
TO MY MOTHER, WHO STILL
HAS TROUBLE PRONOUNCING
"STATISTICS"

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ON TIES IN TRIPLE COMPARISONS

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This study is concerned with the development of triple comparison models that account for ties among the observations. The generalized Bradley-Terry model for paired comparisons is developed by means of the extreme value distributions of Fisher and Tippett, Gnedenko and others in the framework of response-stimulus. Using the framework developed, a generalized Bradley-Terry model for triple comparisons with ties is developed. Due to its lack of symmetry and reversibility, as well as other undesirable properties, this model is abandoned and the Bradley-Pendergrass model for triple comparisons is generalized so as to account for ties among the observations. This generalization is accomplished by considering a triple comparison to be tacitly composed of three independent paired comparisons. Using the generalized Bradley-Terry paired comparison model for the paired comparison probabilities and conditioning the outcome upon the event that no inconsistent rankings occur, we arrive at a generalized Bradley-Pendergrass model.

An iterative procedure for obtaining the maximum likelihood estimates of the parameters introduced into the model is given and their large sample properties investigated. Tests of the parameters of the generalized Bradley-Pendergrass model based on likelihood ratios are given. The power of the likelihood ratio test for testing the equality of treatment ratings is found and compared with the same test using the Bradley-Pendergrass united model as well as the analysis of variance F tests for balanced complete and incomplete block designs. The asymptotic relative efficiencies of the likelihood ratio test using the generalized Bradley-Pendergrass model to the other three aforementioned tests are also given.

The study includes a numerical example demonstrating the use of the estimators and tests developed for the generalized Bradley-Pendergrass model.

CHAPTER 1

INTRODUCTION

In situations where quantitative measurements of treatment effects are not possible, or perhaps not practical, methods of rank analysis are frequently used to analyze the results of these experiments. Problems of statistical analysis based on ranks arise in experimental designs paralleling those used in the classical analysis of variance approach. Much has been done in the area of complete block designs; however, in the area of balanced incomplete block designs, with the exception of the work done by Durbin [8],[1] rank analysis techniques have been limited to the cases of paired and triple comparisons.

In a paired or triple comparison in which a panel of judges ranks two or three treatments at a time on the basis of some quality, a judge may not be able to express a real preference among several of the treatments to be judged. The inability of a judge to state a clear and decisive preference may be due to the fact that his sense of perception is not keen enough to detect the existing difference or perhaps due to the fact that the treatments do not differ in the quality judged.

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[1] Numbers in brackets identify references in the bibliography.

Most models postulated do not allow for the possibility of a judge declaring a tie between two or among three treatments. If ties do occur, in practice they have been (a) completely ignored, (b) equally divided among the tied members, or (c) divided randomly among the tied members. In disregaring ties, however, full use of the information contained in the data is not being made.

Glenn and David [10] have proposed a modification of the Thurstone-Mosteller model for paired comparisons, while Rao and Kupper [21] have proposed a modification of the Bradley-Terry model for paired comparisons, each modification making provision for tied observations. In this paper we generalize the Bradley-Terry [4] and the Bradley-Pendergrass [20] models for triple comparisons so as to make provision for ties.

In Chapter 2 generalized Bradley-Terry models for paired and triple comparisons are derived, using limit theorems. The paired comparison experiment is put into the framework of the stimulus-response phenomena and, following the approach of Thompson and Singh [24], the generalized Bradley-Terry model for paired comparisons is derived, using the limit theorems developed by Fisher and Tippett [9], Gnedenko [11], and others. Using this same framework, a generalized Bradley-Terry model for triple comparisons is developed.

Since the generalized Bradley-Terry model derived using limit theorems lacks the properties of symmetry and reversibility, as well as being unwieldy in form, in Chapter 3 the Bradley-Pendergrass model for triple comparisons is generalized to account for the occurrence of ties. Since this model overcomes the objections to the generalized Bradley-Terry model, it is further explored.

In Chapter 4 the parameters underlying the generalized Bradley-Pendergrass model are estimated, using the method of maximum likelihood; in addition, likelihood ratio tests of these parameters are presented.

The case of n replicates of the total experiment, as well as the case of g groups of replicates with n_u replicates in the u^{th} group $(u = 1,2,\ldots,g)$, is considered.

Large sample properties of the estimates and tests proposed in Chapter 4 are given in Chapter 5. The limiting distribution of the estimators is developed; the power of the likelihood ratio test of equality of treatment ratings for the generalized Bradley-Pendergrass model for triple comparisons is compared to that for the Bradley-Pendergrass model and the analysis of variance F tests for balanced complete and incomplete block designs. The asymptotic relative efficiency of the likelihood ratio test, using the generalized Bradley-Pendergrass model, to the other three tests just mentioned are also given.

A numerical example demonstrating the use of the estimates and tests developed for the generalized Bradley-Pendergrass model is given in Chapter 5.

CHAPTER 2

THE USE OF LIMIT THEOREMS IN PAIRED AND TRIPLE COMPARISON GENERALIZED MODEL BUILDING

2.1 Introduction

The paired and triple comparison experimental arrangements are common in the behavioral sciences. In a paired comparison experiment, a subject is exposed to two stimuli and is asked to report the greater of the two stimuli he experiences. In a triple comparison, on the other hand, the subject is exposed to three stimuli and is asked to report the relative magnitude of the resulting experienced sensations. Most existing models do not allow for the occurrence of ties in the ranking of the stimuli by a subject. In this chapter we shall consider probabilistic models derived using limit theorems for both paired and triple comparisons which allow for ties between two or among three stimuli if the sensations evoked by these stimuli are nondistinguishable to the subject.

2.2 Background for Paired Comparisons

Specific forms have been proposed for $P[X_1 \rightarrow X_2]$, the probability that a subject reports stimulus X_1 as greater than X_2 in a pairwise comparison experiment. Thurstone [25] advanced the law of comparative judgments. The sensations y_1 and y_2 evoked in an individual by a pair of stimuli, X_1 and X_2, were assumed normal and independent with means μ_1 and μ_2 and common variance σ^2. Mosteller [16] noted that the assumption of independence was not essential and could be relaxed to the condition of equal correlations for all pairs of stimuli. Their model may be summarized by writing

$$P[X_1 \to X_2] = (2\pi)^{-1/2} \int_{-(\mu_1-\mu_2)/\sigma}^{\infty} \exp - \frac{1}{2} t^2 \, dt. \qquad (2.2.1)$$

Bradley and Terry [3] proposed an alternative to the law of comparative judgments, postulating that

$$P[X_1 \to X_2] = \frac{1}{4} \int_{-(\ln \pi_1 - \ln \pi_2)}^{\infty} \operatorname{sech}^2 \left(\frac{t}{2}\right) dt, \qquad (2.2.2)$$

where $(\ln \pi_1 - \ln \pi_2)$ could be regarded as a location parameter for the variate $(y_1 - y_2)$. Glenn and David [10] have proposed a modification of the Thurstone-Mosteller model that makes provision for a tie between two stimuli by replacing the normal distribution of the Thurstone-Mosteller model by an "angular" distribution, postulating that a subject will declare a tie when the difference between the two responses lies below a certain threshold. Essentially, Glenn and David postulate the existence of a parameter τ and an interval of length 2τ around the true mean value of the difference in experienced sensation, S_1-S_2, such that

$$\begin{aligned} P[X_1 \to X_2] &= P[y_1-y_2 > \tau], \\ P[X_1 = X_2] &= P[\,|y_1-y_2| < \tau], \qquad (2.2.3) \\ P[X_2 \to X_1] &= P[y_1-y_2 < -\tau], \end{aligned}$$

where for $Z = Y_1 - Y_2$,

$$F_Z(a) = (2\pi)^{-1/2} \int_{-a-(S_1-S_2)}^{\infty} \exp -\frac{1}{2} t^2 \, dt. \tag{2.2.4}$$

Hence, before being manipulated and transformed, the Glenn-David probabilities correspond to

$$\begin{aligned} P[X_1 \rightarrow X_2] &= F(\tau), \\ P[X_1 = X_2] &= F(\tau) - F(-\tau), \\ P[X_2 \rightarrow X_1] &= 1 - F(-\tau). \end{aligned} \tag{2.2.5}$$

Rao and Kupper [21] have proposed a generalization of the Bradley-Terry model, using an approach similar to that of Glenn and David. They introduce a threshold parameter into the Bradley-Terry model to account for ties, showing that

$$P[X_1 \rightarrow X_2] = \frac{1}{4} \int_{-(v_1-v_2)+\eta}^{\infty} \operatorname{sech}^2 \left(\frac{t}{2}\right) dt = \frac{\pi_1}{\pi_1+\theta\pi_2},$$

$$P[X_1 = X_2] = \frac{1}{4} \int_{-(v_1-v_2)-\eta}^{-(v_1-v_2)+\eta} \operatorname{sech}^2 \left(\frac{t}{2}\right) dt = \frac{\pi_1\pi_2(\theta^2-1)}{(\pi_1+\theta\pi_2)(\pi_2+\theta\pi_1)},$$

$$P[X_2 \rightarrow X_1] = \frac{1}{4} \int_{-(v_1-v_2)-\eta}^{\infty} \operatorname{sech}^2 \left(\frac{t}{2}\right) dt = \frac{\pi_2}{\pi_2+\theta\pi_1}, \tag{2.2.6}$$

for $\theta = \exp \eta$ and $\pi_i = \exp v_i$, $i = 1,2$.

Thompson and Singh [24] have shown by the use of limit theorems that the Thurstone-Mosteller model arises quite naturally when experienced sensation is either the sum or some quantile of a large number of responses, while the Bradley-Terry model arises when experienced sensation is the maximum of a large number of responses.

2.3 Statistical Interpretation of the Stimulus-Response Phenomena

In order to present another way in which the Rao-Kupper generalization of the Bradley-Terry model can arise, let us review the psychological framework proposed by Thompson and Singh [24]. Sensory perception begins with the activation of a subject's receptors by a stimulus. A large number of signals are then transmitted from individual receptors to the brain where they are synthesized into an overall intensity of the stimulus. Intensity of a given stimulus is thought to depend upon the number of receptors activated and the frequency of neural activity within each receptor. To account for the fact that a subject reports the more intense stimulus as the lesser stimulus, a random error in the transmission system is postulated. This random error appears as the relative frequency with which a subject reports stimulus X_1 as greater than stimulus X_2. This relative frequency approximates $P[X_1 \rightarrow X_2]$. We shall assume that a subject receiving stimulus X experiences a sensation y, where y is real-valued and subject to chance, and that the experienced sensations y_1 and y_2 corresponding to stimuli X_1 and X_2 satisfy

$$P[X_1 \rightarrow X_2] = P[\frac{y_1}{y_2} > C]$$

for C a real number. The sensation parameter, denoted by Y, is defined as the mode of the distribution of experienced sensation, y.

2.4 The Use of Limit Theorems in Paired Comparisons

Let $u_1, u_2, \ldots, u_m$ denote signals transmitted to the brain by m sense receptors in response to stimulus X, where m is large, but fixed. The individual signals are considered to be random variables which, when combined psychophysically, form the experienced sensation y_m. We shall assume that there exist sequences of constants, $\{\mu'_m\}$ and $\{\sigma'_m\}$ with $\sigma'_m > 0$, and a distribution function H(y) such that

$$\lim_{m \to \infty} P\left[\frac{y_m - \mu'_m}{\sigma'_m} \leq y\right] = H(y).$$

Stevens [22] reports that there may be two basic mechanisms of discrimination: (1) an increase in intensity of a stimulus results when neural excitation is added to excitation already present, whereas (2) a change in the quality of a stimulus results when there is a change in the distribution of excitation, i.e., when new excitation is substituted for old. This suggests two of the three models considered by Thompson and Singh.

1. Experienced sensation, y, is the sum (or average) of a large number of signals transmitted by individual receptors.

2. Experienced sensation, y, is the median or some quantile of a large number of transmitted signals.
3. Experienced sensation, y, is the maximum of a large number of transmitted signals.

Models 1 and 2 lead to asymptotic normal distributions for experienced sensation, while model 3 leads to the extreme value distributions developed by Fisher and Tippett [9], Mises [15], Gnedenko [11], Gumbel [12] and others. Their research is summarized in the following theorem.

Theorem 2.4.1.

Suppose $(u_1, u_2, \ldots)$ are independent random variables with a common distribution function F. If, for $y_m = \max\{u_1, u_2, \ldots, u_m\}$, there exist sequences of constants, $\{\mu'_m\}$ and $\{\sigma'_m\}$, $\sigma'_m > 0$, such that

$$\lim_{m \to \infty} P\left[\frac{y_m - \mu'_m}{\sigma'_m} \le y\right] = H(y) \tag{2.4.1}$$

exists and is a non-degenerate distribution function, then $H(y) = \Psi_i(ay+b)$, $a > 0$ for one of the three following functions:

$$\begin{aligned}
\Psi_1(y) &= \exp(-e^{-y}) , && -\infty < y < \infty \\
\Psi_2(y) &= \begin{cases} 0 & y \le 0 \\ \exp(-y^{-\alpha}) , & y > 0 \end{cases} && \alpha > 0 \\
\Psi_3(y) &= \begin{cases} \exp -(-y)^{\alpha} & y < 0 \\ 1 & y \ge 0 \end{cases} . && \alpha > 0
\end{aligned} \tag{2.4.2}$$

(Proof is omitted.)

Thus, if y_m has a non-degenerate limiting distribution, we may write approximately for large m,

$$P[y_m \leq y] \approx \Psi_i \left[\frac{y-\mu_m}{\sigma_m}\right] \tag{2.4.3}$$

where $\sigma_m = \sigma'_m/a$, $\mu_m = \mu'_m - b\sigma'_m/a$ and $\Psi_i(y)$ one of the three functions of theorem 2.4.1. The initial distribution function, F, is said to be of exponential, Cauchy, or limited type according as the limit distribution is Ψ_1, Ψ_2 or Ψ_3.

Thompson and Singh [24] interpret the parameters μ_m and σ_m in terms of the limiting distribution. $\Psi_1[(y-\mu_m)/\sigma_m]$ has mode μ_m, while σ_m is a scale parameter. For an initial distribution of the Cauchy type, $\mu_m = 0$ for all m; hence the limit will be of the form $\Psi_2(y/\sigma_m;\alpha)$ which has mode $\sigma_m(\alpha/\alpha+1)^{1/\alpha}$. If F is of the limited type, there exists a finite number ω, such that $F(\omega) = 1$, but for each $\epsilon > 0$, $F(\omega-\epsilon) < 1$; thus $\mu_m = \omega$ for all m. The mode of $\Psi_3[(y-\omega)/\sigma_m]$ is ω for $0 < \alpha \leq 1$ and $\omega - \sigma_m[(\alpha-1)/\alpha]^{1/\alpha}$ for $\alpha > 1$.

The following theorem will be useful in investigating the limit distributions.

Theorem 2.4.2.

If $m \to \infty$, $n \to \infty$ such that $\sigma_{2n}/\sigma_{1m} \to \lambda$ and if the bivariate distribution of the random variables

$$\frac{Y_{1m} - \mu_{1m}}{\sigma_{1m}}, \quad \frac{Y_{2n} - \mu_{1n}}{\sigma_{2n}}$$

converges to that of Z_1, Z_2, then for ρ real,

$$P\left[\frac{(Y_{1m}-\rho\ Y_{2n})-(\mu_{1m}-\rho\ \mu_{2n})}{(\sigma_{1m}^2+\rho^2\ \sigma_{2n}^2)^{1/2}}\leq x\right]$$

$$\to P\left[\frac{Z_1-\rho\ \lambda\ Z_2}{(1+\rho^2\lambda^2)^{1/2}}\leq x\right].$$

Proof of theorem 2.4.2. Write

$$\frac{(Y_{1m}-\rho\ Y_{2n})-(\mu_{1m}-\rho\ \mu_{2n})}{(\sigma_{1m}^2+\rho^2\ \sigma_{2n}^2)^{1/2}}=\frac{Y_{1m}-\mu_{1m}}{\sigma_{1m}\left(1+\rho^2\ \dfrac{\sigma_{2n}^2}{\sigma_{1m}^2}\right)^{1/2}}-\frac{Y_{2n}-\mu_{2n}}{\sigma_{2n}\left(1+\dfrac{\sigma_{1m}^2}{\rho^2\ \sigma_{2n}^2}\right)^{1/2}}.$$

Let $\varphi_{mn}(t_1,t_2)$ be the joint characteristic function of the standardized random variables

$$\frac{Y_{1m}-\mu_{1m}}{\sigma_{1m}}\text{ and }\frac{Y_{2n}-\mu_{2n}}{\sigma_{2n}}.$$

$\varphi_{mn}(t_1,t_2)$ converges uniformly to $\varphi(t_1,t_2)$, the joint characteristic function of Y_1 and Y_2. Therefore the joint characteristic function of

$$\frac{(Y_{1m}-\rho\ Y_{2n})-(\mu_{1m}-\rho\ \mu_{2n})}{(\sigma_{1m}^2+\rho\ \sigma_{2n}^2)^{1/2}},$$

given as

$$\varphi_{mn}\left(t_1\left[1+\rho^2\ \frac{\sigma_{2n}^2}{\sigma_{1m}^2}\right]^{-1/2},\ -t_1\left[1+\frac{\sigma_{1m}^2}{\rho^2\sigma_{2n}^2}\right]^{-1/2}\right),$$

converges uniformly to

$$\varphi\left(t_1[1+\rho^2\lambda^2]^{-1/2}\ ,\ -t_1[1+\rho^{-2}\lambda^{-2}]^{-1/2}\right),$$

which is the characteristic function of the random variable

$$(Z_1 - \rho \lambda Z_2)/(1+ \rho^2 \lambda^2)^{1/2} .$$

Hence, the theorem is proved.

In the investigation of models 1 and 2 for the untied case, Thompson and Singh arrived at the Thurstone-Mosteller model, while model 3 led to the Bradley-Terry model. We restrict our attention to model 3, where experienced sensation, y_m, is the maximum of the m signals $(u_1, u_2, \ldots, u_m)$, and derive the generalized Bradley-Terry model for paired comparisons as given by Rao and Kupper [21].

2.5 The Asymptotic Derivation of the Generalized Bradley-Terry Model for Paired Comparisons

The Exponential Type Distribution

Let y_{1m} be the experienced sensation defined as $\max(u_1, u_2, \ldots, u_m)$, where $\{u_1, u_2, \ldots\}$ is a sequence of independent, identically distributed random variables with common distribution function F, while $y_{2n} = \max(v_1, v_2, \ldots, v_n)$ where $\{v_1, v_2, \ldots\}$ is a sequence of independent identically distributed random variables with common distribution function G.

In the comparison of two stimuli, X_1 and X_2, with F and G exponential type distributions, we postulate the existence of a parameter, η, whereby a subject will report

$$
\begin{array}{lll}
X_1 \rightarrow X_2 & \text{if} & y_{1m} - y_{2n} > \eta , \\
X_1 = X_2 & \text{if} & |y_{1m} - y_{2n}| < \eta \\
\end{array} \qquad (2.5.1)
$$

and

$$X_2 \rightarrow X_1 \quad \text{if} \quad y_{1m} - y_{2n} < -\eta .$$

Hence η is a threshold parameter, since the subject reports no difference between stimuli X_1 and X_2 unless the difference between the experienced sensations exceeds η in absolute value.

Now there exist sequences of constants, $\{(\mu_{im}, \sigma_{im})\}$ $(i=1,2)$, such that

$$\lim_{m \to \infty} P\left[\frac{y_{1m}-\mu_{1m}}{\sigma_{1m}} \leq y\right] = \Psi_1(y)$$

and

$$\lim_{n \to \infty} P\left[\frac{y_{2n}-\mu_{2n}}{\sigma_{2n}} \leq y\right] = \Psi_1(y) ,$$

where

$$\Psi_1(y) = \exp(-e^{-y}) \qquad -\infty < y < \infty . \qquad (2.5.2)$$

If m and n tend to ∞ such that $\sigma_{2n}/\sigma_{1m} \rightarrow \lambda$, it is easily shown that

$$P\left[\frac{(y_{1m}-y_{2n}) - (\mu_{1m}-\mu_{2n})}{(\sigma_{1m}^2 + \sigma_{2n}^2)^{1/2}} \leq z\right]$$

$$\rightarrow \int_0^\infty \exp - [wt^\lambda + t]\, dt \qquad (2.5.3)$$

with $w = \exp\{-z(1+\lambda^2)^{1/2}\}$ which for $\lambda = 1$ simplifies to

$$[1 + \exp(\sqrt{2}\, z)]^{-1}. \qquad (2.5.4)$$

Now consider $P[X_1 \rightarrow X_2]$, which is given as

$$P[y_{1m}-y_{2n} > \eta] = 1 - P[y_{1m}-y_{2n} < \eta]$$

$$= 1 - P\left[\frac{(y_{1m}-y_{2n}) - (\mu_{1m}-\mu_{2n})}{(\sigma_{1m}^2 + \sigma_{2n}^2)^{1/2}} \leq \frac{\eta - (\mu_{1m}-\mu_{2n})}{(\sigma_{1m}^2+\sigma_{2n}^2)^{1/2}}\right] . \tag{2.5.5}$$

For large values of m and n with $\lambda = 1$, we can write

$$P[y_{1m}-y_{2n} > \eta] \approx 1 - \left[1 + \exp \sqrt{2}\left(\frac{\eta - (\mu_{1m}-\mu_{2n})}{(\sigma_{1m}^2+\sigma_{2n}^2)^{1/2}}\right)\right]^{-1} , \tag{2.5.6}$$

so that for $\theta = \exp \sqrt{2}\, \eta \Big/ (\sigma_{1m}^2+\sigma_{2n}^2)^{1/2}$ and π_1 and π_2 satisfying the equation

$$\log\left(\frac{\pi_1}{\pi_2}\right) = \frac{\mu_{1m} - \mu_{2n}}{[\frac{1}{2}(\sigma_{1m}^2+\sigma_{2n}^2)]^{1/2}} , \tag{2.5.7}$$

we get from (2.5.5)

$$P[y_{1m}-y_{2n} > \eta] \approx 1 - \frac{\theta\, \pi_2}{\pi_1 + \theta\pi_2} ,$$

or

$$P[X_1 \rightarrow X_2] \approx \frac{\pi_1}{\pi_1 + \theta\pi_2} . \tag{2.5.8}$$

In a similar manner, we can show that for large m and n,

$$P[X_2 \rightarrow X_1] \approx \frac{\pi_2}{\pi_2 + \theta\pi_1} ,$$

while

$$P[X_1 = X_2] \approx \frac{\pi_1\pi_2(\theta^2-1)}{(\pi_1+\theta\pi_2)(\pi_2+\theta\pi_1)} . \tag{2.5.9}$$

Equations (2.5.2) imply that asymptotically, y_{1m} and y_{2n} have distribution functions

$$\Psi_1\left(\frac{y_1-\mu_{1m}}{\sigma_{1m}}\right) \text{ and } \Psi_1\left(\frac{y_2-\mu_{2n}}{\sigma_{2n}}\right),$$

which have modes μ_{1m} and μ_{2n}, respectively. Therefore, for this model, Y_1, the sensation due to stimulus X_1, was defined to be μ_{1m}, the mode of y_{1m}. Similarly, Y_2, the sensation due to X_2, is given as μ_{2n}.

Writing

$$\alpha = [\tfrac{1}{2}(\sigma_{1m}^2 + \sigma_{2n}^2)]^{1/2}$$

in equation (2.5.7), we obtain

$$\log\left(\frac{\pi_1}{\pi_2}\right) = \frac{Y_1 - Y_2}{\alpha}$$

$$\frac{\pi_1}{\pi_2} = \exp\frac{Y_1}{\alpha} \cdot \exp\frac{-Y_2}{\alpha}$$

or

$$\frac{\pi_1}{\exp\frac{Y_1}{\alpha}} = \frac{\pi_2}{\exp\frac{Y_2}{\alpha}}, \tag{2.5.10}$$

which says $\pi_i = k \exp\frac{Y_i}{\alpha} \qquad i = 1,2.$

Hence, for initial exponential type distributions, with $\lambda = 1$, we have obtained the generalized Bradley-Terry model:

$$P[X_1 \to X_2] \approx \left[1 + \theta \exp - \frac{(Y_1 - Y_2)}{\alpha}\right]^{-1}. \tag{2.5.11}$$

Equation (2.5.11) provides a psychological interpretation of the generalized Bradley-Terry model in terms of the sensation parameters, Y_1 and Y_2.

The assumption that $\lambda = 1$ implies that the experienced sensations have asymptotically equal dispersions but different modal values.

The Cauchy Type Distribution

As for the exponential type distribution, we assume that y_{1m} and y_{2n} are each the maximum of a set of m and n independent, identically distributed random variables having distribution functions F and G, respectively; however, we now consider F and G to be Cauchy type distributions. For two Cauchy type distributions the sequences $\{\mu_{im}\}$ (i=1,2) can be taken to be sequences of zeros. Therefore, there exist sequences, $\{\sigma_{1m}\}$ and $\{\sigma_{2n}\}$, such that

$$\lim_{m \to \infty} P\left[\frac{y_{1m}}{\sigma_{1m}} \leq y\right] = \Psi_2(y;\alpha_1)$$

and

$$\lim_{n \to \infty} P\left[\frac{y_{2n}}{\sigma_{2n}} \leq y\right] = \Psi_2(y;\alpha_2) \quad ,$$

for

$$\Psi_2(y;\alpha_i) = \begin{cases} 0 & y \leq 0 \quad \alpha_i > 0 \quad (i=1,2) \\ \exp -y^{-\alpha_i} & y > 0 \end{cases} \quad . \qquad (2.5.12)$$

Since the measures of central tendency are known, we propose that, as in the case of known means, that the ratio of the experienced sensations y_{1m} and y_{2n} be investigated. We postulate the existence of a threshold parameter, η, whereby a subject reports

$$\begin{aligned} &X_1 \rightarrow X_2 \quad \text{if} \quad \log y_{1m} - \log y_{2n} > \eta \,, \\ &X_1 = X_2 \quad \text{if} \quad |\log y_{1m} - \log y_{2n}| < \eta \\ \text{and} \quad &X_2 \rightarrow X_1 \quad \text{if} \quad \log y_{1m} - \log y_{2n} < -\eta \,. \end{aligned} \tag{2.5.13}$$

Let us evaluate $P[X_2 \rightarrow X_1]$ by considering

$$\begin{aligned} P[\log y_{1m} - \log y_{2n} < -\eta] &= P[y_{1m}/y_{2n} < e^{-\eta}] \\ &= P[y_{1m} - e^{-\eta} y_{2n} < 0] \\ &= P\left[\frac{y_{1m} - e^{-\eta} y_{2n}}{(\sigma_{1m}^2 + e^{-2\eta}\sigma_{2n}^2)^{1/2}} < 0\right] . \end{aligned} \tag{2.5.14}$$

From theorem 2.4.2, with $\rho = e^{-\eta}$, we have

$$\lim_{m,n\rightarrow\infty} P\left[\frac{y_{1m} - y_{2n}}{(\sigma_{1m}^2 + \rho^2 \sigma_{2n}^2)^{1/2}} < z\right]$$

$$\rightarrow \int_{-\frac{z[1+\rho^2\lambda^2]^{1/2}}{\rho\lambda}}^{\infty} \Psi_{\alpha_2}\left[\rho\lambda t + z(1+\rho^2\lambda^2)^{1/2}\right] d\,\Psi_{\alpha_1}(t), \tag{2.5.15}$$

which for $w = z(1 + \rho^2\lambda^2)^{1/2}$ becomes

$$\int_0^{[-\frac{\omega}{\rho\lambda}]^{-\alpha_2}} \exp - [(w + \rho\lambda u^{-1/\alpha_2})^{-\alpha_1} + u] \, du \,. \tag{2.5.16}$$

For $z = 0$ and $\alpha_1 = \alpha_2 = \alpha$, the integral in (2.5.15) and (2.5.16) simplifies, giving

$$\lim_{m,n\to\infty} P\left[\frac{y_{1m} - y_{2n}}{(\sigma_{1m}^2+\rho^2\sigma_{2n}^2)^{1/2}} < 0\right] = \int_0^\infty \exp - u\,[(\lambda\rho)^{-\alpha} + 1]\,du$$

$$= \frac{1}{1 + (\lambda\rho)^{-\alpha}}\,. \qquad (2.5.17)$$

As an approximation when m and n are large, one can write

$$\lambda \approx \frac{\sigma_{2n}}{\sigma_{1m}}\,,$$

and hence

$$P[X_2 \to X_1] \approx \frac{1}{1 + \rho^{-\alpha}\cdot\left(\frac{\sigma_{1m}}{\sigma_{2n}}\right)^{\alpha}}$$

$$= \frac{\sigma_{2n}^{\alpha}}{\sigma_{2n}^{\alpha} + \rho^{-\alpha}\,\sigma_{1m}^{\alpha}}\,. \qquad (2.5.18)$$

Now for $\pi_1 \propto \sigma_{1m}^{\alpha}$, $\pi_2 \propto \sigma_{2n}^{\alpha}$ and $\theta \propto \rho^{-\alpha}$, we have the form of the generalized Bradley-Terry model given as

$$P[X_2 \to X_1] \approx \frac{\pi_2}{\pi_2 + \theta\pi_1}\,. \qquad (2.5.19)$$

From (2.5.18) and the definition of true sensation as the mode of y, we have

$$Y_1 = \sigma_{1m}\left(\frac{\alpha}{\alpha+1}\right)^{1/\alpha} \text{ and } Y_2 = \sigma_{2n}\left(\frac{\alpha}{\alpha+1}\right)^{1/\alpha}, \qquad (2.5.20)$$

which yields

$$P[X_2 \rightarrow X_1] = \left[1 + \theta \left(\frac{Y_1}{Y_2}\right)^{\alpha}\right]^{-1} . \qquad (2.5.21)$$

It is easily shown that

$$P[X_1 \rightarrow X_2] \approx \frac{\pi_1}{\pi_1 + \theta\pi_2}$$

and

$$P[X_1 = X_2] \approx \frac{\pi_1\pi_2(\theta^2-1)}{(\pi_1+\theta\pi_2)(\pi_2+\theta\pi_1)} .$$

The Limited Type Distribution

Suppose y_{1m} and y_{2n} are each the maximum of a set of independent, identically distributed random variables having distribution functions F and G, respectively, where F and G are limited type distributions. Then there exist real numbers, ω_1, ω_2, and sequences, $\{\sigma_{1m}\}$, $\{\sigma_{2n}\}$, such that

$$\lim_{m\rightarrow\infty} P\left[\frac{y_{1m} - \omega_1}{\sigma_{1m}} \leq y\right] = \Psi_3(y;\alpha_1)$$

and

$$\lim_{n\rightarrow\infty} P\left[\frac{y_{2m} - \omega_2}{\sigma_{2n}} \leq y\right] = \Psi_3(y;\alpha_2) ,$$

for

$$\Psi_3(y;\alpha_i) = \begin{cases} \exp - (-y)^{\alpha_i} & y \leq 0 \\ 1 & y > 0 \end{cases} \qquad \alpha_i > 0 \quad (i=1,2). \qquad (2.5.22)$$

When comparing two random variables with known measures of central tendency and unknown measures of dispersion, it is natural to compare the ratio of the random variables, each corrected for its measure of central tendency. Although the quantities ω_1 and ω_2 will be unknown in practice, we propose to show later in this chapter that no changes in methodology will be encountered if one has consistent estimates of ω_1 and ω_2. Hence we now proceed, assuming ω_1 and ω_2 known, to investigate the ratio $(y_{1m}-\omega_1/(y_{2n}-\omega_2)$.

We postulate the existence of a threshold parameter, η, whereby the subject will report

$$X_1 \rightarrow X_2 \quad \text{if} \quad (y_{1m}-\omega_1)/(y_{2n}-\omega_2) < e^{-\eta} \ ,$$

$$X_1 = X_2 \quad \text{if} \quad e^{-\eta} < (y_{1n}-\omega_1)/(y_{2n}-\omega_2) < e^{\eta}$$

$$\text{and} \quad X_2 \rightarrow X_1 \quad \text{if} \quad (y_{1m}-\omega_1)/(y_{2n}-\omega_2) > e^{\eta} \ .$$

Recall that y_{1m} and y_{2n} are limited to the right by ω_1 and ω_2, respectively, and therefore $(y_{1m}-\omega_1)$ and $(y_{2n}-\omega_2)$ are both negative quantities. Now consider

$$\begin{aligned} P[X_1 \rightarrow X_2] &= P\left[\frac{(y_{1m}-\omega_1)}{(y_{2n}-\omega_2)} < e^{-\eta}\right] \\ &= P\left[(y_{1m}-\omega_1) > e^{-\eta}\,(y_{2n}-\omega_2)\right] \\ &= P\left[(y_{1m}-e^{-\eta}\,y_{2n}) - (\omega_1-e^{-\eta}\,\omega_2) > 0\right] \ , \end{aligned} \tag{2.5.23}$$

which for $\rho = e^{-\eta}$ can be written as

$$1 - P\left[\frac{(y_{1m}-\rho y_{2n}) - (\omega_1-\rho\omega_2)}{(\sigma_{1m}^2+\rho^2\sigma_{2n}^2)^{1/2}} < 0\right] . \tag{2.5.24}$$

By theorem 2.4.2, with $w = z(1 + \rho^2\lambda^2)^{1/2}$ and $\rho = e^{-\eta}$, it can be deduced that

$$P\left[\frac{(y_{1m}-\rho y_{2n})-(\omega_1-\rho\omega_2)}{(\sigma_{1m}^2+\rho^2\sigma_{2n}^2)^{1/2}} < z\right] \rightarrow \int_{-\infty}^{-\frac{w}{\lambda\rho}} \Psi_3(\lambda\rho t + w;\alpha_1)\, d\Psi_3(t;\alpha_2) - \int_{-\frac{w}{\lambda\rho}}^{0} d\Psi_3(t;\alpha_2) . \tag{2.5.25}$$

The right-hand side of (2.5.25) can be written as

$$\int_{-\infty}^{-\frac{w}{\lambda\rho}} \alpha_2(-t)^{\alpha_2-1} \exp\left\{[-(\rho\lambda t+w)]^{-\alpha_1} + (-t)^{\alpha_2}\right\} dt - \int_{-\frac{w}{\rho\lambda}}^{0} \alpha_2(-t)^{\alpha_2-1} \exp[-(-t)^{\alpha_2}]\, dt. \tag{2.5.26}$$

When $z = 0$ and $\alpha_1 = \alpha_2 = \alpha$ in (2.5.26), we obtain

$$P[X_1 \rightarrow X_2] \rightarrow \frac{1}{1+\rho^{\alpha}\lambda^{\alpha}} . \tag{2.5.27}$$

For large values of m and n, writing $\sigma_{2n}/\sigma_{1m} \approx \lambda$ we get

$$P[X_1 \to X_2] \approx 1 - \frac{1}{1 + \rho^{\alpha}\left(\frac{\sigma_{2n}}{\sigma_{1m}}\right)^{\alpha}}$$

$$= 1 - \frac{\rho^{-\alpha}\sigma_{2n}^{-\alpha}}{\rho^{-\alpha}\sigma_{2n}^{-\alpha} + \sigma_{1m}^{-\alpha}}$$

$$= \frac{\sigma_{1m}^{-\alpha}}{\sigma_{1m}^{-\alpha} + \rho^{-\alpha}\sigma_{2n}^{-\alpha}} . \tag{2.5.28}$$

For $\pi_1 \propto \sigma_{1m}^{-\alpha}$, $\pi_2 \propto \sigma_{2n}^{-\alpha}$ and $\theta \propto \rho^{-\alpha}$, (2.5.28) is in the form of the Rao-Kupper generalization of the Bradley-Terry model.

For $0 < \alpha < 1$, we have no psychological interpretation of the model in terms of parameters, but for $\alpha \geq 1$,

$$Y_1 = \omega_1 - \sigma_{1m}(\alpha-1/\alpha)^{1/\alpha} ,$$

$$Y_2 = \omega_2 - \sigma_{2n}(\alpha-1/\alpha)^{1/\alpha} .$$

Therefore, in terms of sensation parameters, we can write

$$P[X_1 \to X_2] \approx \left[1 + \theta\left(\frac{\omega_2 - Y_2}{\omega_1 - Y_1}\right)^{-\alpha}\right]^{-1} . \tag{2.5.29}$$

In a manner similar to that for the exponential and Cauchy type distributions, we find

$$P[X_2 \to X_1] \approx \frac{\pi_2}{\pi_2 + \theta\pi_1}$$

and

$$P[X_1 = X_2] \approx \frac{\pi_1\pi_2(\theta^2 - 1)}{(\pi_1 + \theta\pi_2)(\pi_2 + \theta\pi_1)} .$$

Estimation of the Location Parameter in the Limited Type Distribution

In the development of the generalized Bradley-Terry model for the limited type distribution in this section, we assumed that ω_1 and ω_2, the upper limits of sensation, were known constants. However, in practice these quantities would rarely be known; hence estimates of these parameters are required. No change in preceding methodology is required if a consistent estimator for ω is available.

It is easily verified that the mode of $\Psi_3[(y-\omega)/\sigma_n]$ is ω if $0 < \alpha \leq 1$ and $\omega - \sigma_n(\sigma-1/\alpha)^{1/\alpha}$ if $\alpha > 1$. The sequence $\{\sigma_n\}$ has the property that $\sigma_n > 0$, $\sigma_n \geq \sigma_{n+1}$ and is of order $n^{-1/\alpha}$ (see, for example, Gumbel [12] or Gnedenko [11]). Hence, asymptotically, the mode of the limiting distribution is ω for $\alpha > 0$.

It is reasonable to assume that a subject is able to arrive at an empirical estimate of ω. In fact, Parzen [18] has given a class of estimators for the mode and density function corresponding to a given distribution function which are consistent under certain regularity conditions. Venter [26] also gives two consistent estimators of the mode of a density function which are appealing. For an ordered sample, $(Y_1, Y_2, \ldots, Y_n)$, the mode is estimated by the suitable points in the interval formed by the first and last of those s consecutive Y_i's which are closest together. The mathematics of these estimators is omitted and the reader referred to the two articles, as well as that by Chernoff [6], for further details. It is sufficient here to note that consistent estimators of the mode of a distribution exist.

When using a consistent estimator of the mode, the distribution of $(y-\hat{\omega}_n)/\sigma_n$ is asymptotically the same as that of $(y-\omega)/\sigma_n$ as shown by theorem 2.5.1.

Theorem 2.5.1.

Let F_n denote the distribution function of the random variable $(y-\hat{\omega}_n)/\sigma_n$ and Ψ denote the limiting distribution function of $(y-\omega)/\sigma_n$. If $\hat{\omega}_n$ is a consistent estimator of ω, then F_n converges to Ψ in distribution.

Proof of theorem 2.5.1.

Now

$$\begin{aligned} F_n(z) &= P\left[\frac{y-\hat{\omega}_n}{\sigma_n} \le z\right] \\ &= P\left[\frac{y-\omega}{\sigma_n} - \frac{\hat{\omega}_n-\omega}{\sigma_n} \le z\right] \\ &= P\left[\frac{y-\omega}{\sigma_n} \le z + \frac{\hat{\omega}_n-\omega}{\sigma_n}\right] \\ &= P\left[\frac{y-\omega}{\sigma_n} \le z + \frac{\hat{\omega}_n-\omega}{\sigma_n} , \left|\frac{\hat{\omega}_n-\omega}{\sigma_n}\right| < \epsilon\right] \\ &\quad + P\left[\frac{y-\omega}{\sigma_n} \le z + \frac{\hat{\omega}_n-\omega}{\sigma_n} , \left|\frac{\hat{\omega}_n-\omega}{\sigma_n}\right| \ge \epsilon\right] . \end{aligned} \qquad (2.5.30)$$

Let

$$S_1 = \left[\frac{y-\omega}{\sigma_n} \leq z + \frac{\hat{\omega}_n - \omega}{\sigma_n} \, , \, \left| \frac{\hat{\omega}_n - \omega}{\sigma_n} \right| \geq \epsilon\right] ,$$

$$S_2 = \left[\frac{y-\omega}{\sigma_n} \leq z + \frac{\hat{\omega}_n - \omega}{\sigma_n} \, , \, \left| \frac{\hat{\omega}_n - \omega}{\sigma_n} \right| < \epsilon\right]$$

and

$$S_3 = \left[\left| \frac{\hat{\omega}_n - \omega}{\sigma_n} \right| \geq \epsilon \right] . \tag{2.5.31}$$

For $\hat{\omega}_n$ a consistent estimator of ω, we have

$$\lim_{n \to \infty} P[\, | \hat{\omega}_n - \omega | < \epsilon] = 1 \quad \text{for every} \quad \epsilon > 0 \, . \tag{2.5.32}$$

Noting that σ_n is of order $n^{-1/\alpha}$ for α a parameter in the initial distribution, we write σ_n as $(cn)^{-1/\alpha}$ for $c > 0$; hence, for $c' = c^{\alpha}$ we have

$$P\left[\left| \frac{\hat{\omega}_n - \omega}{\sigma_n} \right| < \epsilon \right] = P\left[c' n^{1/\alpha} \, | \hat{\omega}_n - \omega | < \epsilon\right]$$

$$= P\left[\, | \hat{\omega}_n - \omega | < \frac{\epsilon}{c' n^{1/\alpha}} \right] . \tag{2.5.33}$$

Since (2.5.32) is true for all $\epsilon > 0$, take $\epsilon' = \epsilon / c' n^{1/\alpha}$. Then

$$\lim_{n \to \infty} P\left[\left| \frac{\hat{\omega}_n - \omega}{\sigma_n} \right| < \epsilon \right] = \lim_{n \to \infty} P\left[\, | \hat{\omega}_n - \omega | < \epsilon' \right] = 1. \tag{2.5.34}$$

Therefore, the random variable $(\hat{\omega}_n - \omega)/\alpha_n$ converges to zero in probability, which is to say, $P[S_3] \to 0$. However, S_1 is a subset of S_3 and, therefore, $P[S_1] \to 0$. We can restrict attention to the set S_2. Now

$$P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_n}\right| < \epsilon\right]$$

$$\leq P[S_2] \leq$$

$$P\left[\frac{y-\omega}{\sigma_n} \leq z + \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_n}\right| < \epsilon\right]. \tag{2.5.35}$$

But

$$P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_n}\right| < \epsilon\right] =$$

$$P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon\right] - P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_n}\right| \geq \epsilon\right].$$

Therefore,

$$\lim_{n\to\infty} P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_n}\right| < \epsilon\right] = P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon\right]. \tag{2.5.36}$$

In like manner,

$$\lim_{n\to\infty} P\left[\frac{y-\omega}{\sigma_n} \leq z + \epsilon,\ \left|\frac{\hat{\omega}_n - \omega}{\sigma_m}\right| < \epsilon\right] = P\left[\frac{y-\omega}{\sigma_n} \leq z + \epsilon\right]. \tag{2.5.37}$$

From (2.5.35), (2.5.36) and (2.5.37), we find

$$\lim_{n\to\infty} P\left[\frac{y-\omega}{\sigma_n} \leq z - \epsilon\right] \leq \lim_{n\to\infty} P[S_3] \leq \lim_{n\to\infty} P\left[\frac{y-\omega}{\sigma_n} \leq z + \epsilon\right]. \tag{2.5.38}$$

Since (2.5.38) is true for all $\epsilon > 0$, we can write

$$\Psi(z) \leq \lim_{n\to\infty} F_n(z) \leq \Psi(z)$$

or

$$\lim_{n\to\infty} F_n(z) = \Psi(z)\ . \tag{2.5.39}$$

Hence, we have shown that the random variable $(y-\hat{\omega}_n)/\sigma_n$ has the same asymptotic distribution as $(y-\omega)\sigma_n$; therefore, the theorem is proved.

Writing $\hat{Z}_n = (y-\hat{\omega}_n)/\sigma_n$ and $Z_n = (y-\omega)/\sigma_n$, we have shown that if Z_n converges to the limit random variable Z, then $\hat{Z}_n$ also converges to Z. Mann and Wald [14] in theorem 5 of their paper state that if $\underline{X}_n$ converges in distribution to $\underline{X}$, and $g(\underline{x})$ is a Borel measurable function with the set of discontinuity points of $g(\underline{x})$ closed and of measure zero, then $g(\underline{X}_n)$ converges to $g(\underline{X})$ in distribution. If we define

$$g(\hat{z}_1, \hat{z}_2) = \frac{y_1 - \hat{\omega}_1}{y_2 - \hat{\omega}_2} ,$$

then

$$\frac{y_1 - \hat{\omega}_1}{y_2 - \hat{\omega}_2} \overset{D}{\rightarrow} Z',$$

where Z' is the limiting random variable defined by

$$\frac{y_1 - \omega_1}{y_2 - \omega_2} \overset{D}{\rightarrow} Z' .$$

We have shown, then, that provided consistent estimators of ω_1 and ω_2 exist, a subject can base his decision on either the random variable $(y_1 - \omega_1)/(y_2 - \omega_2)$ or the random variable $(y_1 - \hat{\omega}_1)/(y_2 - \hat{\omega}_2)$ when the number of signals resulting from each stimulus is large.

2.6. The Use of Limit Theorems in Triple Comparisons

In a preliminary report concerning the rank analysis of triple comparison data involving t treatments or items, Bradley and Terry [4] proposed a mathematical model by postulating the existence of parameters, $\pi_1, \pi_2, \ldots, \pi_t$, satisfying the conditions $\pi_i \geq 0$ $(i = 1,2,\ldots,t)$, $\sum_{i=1}^{t} \pi_i = 1$, where the probability that treatment i is preferred to j and k and treatment j is preferred to k is given as

$$P[i \rightarrow j \rightarrow k] = \frac{\pi_i \pi_j}{(\pi_i+\pi_j+\pi_k)(\pi_j+\pi_k)} \; . \tag{2.6.1}$$

Bradley [2] points out that this model is consistent with that proposed by Lehmann [13] under the simplifying assumption that the random variables, say y_i, y_j and y_k, are independent with distribution functions F_i, F_j and F_k, respectively. Then

$$P[i \rightarrow j \rightarrow k] = \iiint_{y_i>y_j>y_k} dF_i(y_i)\; dF_j(y_j)\; dF_k(y_k)$$

$$= \frac{\pi_i \pi_j}{(\pi_i+\pi_j+\pi_k)(\pi_j+\pi_k)} \quad , \tag{2.6.2}$$

where π_i, π_j and π_k are parameters of the distributions, F_i, F_j and F_k.

Bradley and Pendergrass [20] considered the model given in (2.6.1) and (2.6.2), but for reasons of symmetry and reversibility preferred to work with model given as

$$P[i \rightarrow j \rightarrow k] = \frac{\pi_i^2 \pi_j}{\pi_i^2(\pi_j+\pi_k) + \pi_j^2(\pi_i+\pi_k) + \pi_k^2(\pi_i+\pi_j)} \; . \tag{2.6.3}$$

The generalization of the model given in (2.6.3) to account for ties in the data is developed in Chapter 3. In the remainder of this chapter, we shall investigate the generalization of the model given in (2.6.1) and (2.6.2) to account for the presence of ties in the data. As noted in Section 2.5, when considering a model in the psychological context of stimulus and sensation, the generalized Bradley-Terry model resulted when experienced sensation was taken to be the maximum of the signals sent by m sense receptors. For this reason, we too shall take experienced sensation to be the maximum signal. We shall use the same definitions and assumptions given in Sections 2.3 and 2.4 with obvious extensions to account for three stimuli and three sensations considered simultaneously.

In ranking the stimuli X_i, X_j and X_k, we assume that a subject arrives at his total ranking by comparing the three combinations (i,j), (i,k) and (j,k) in the form of ratios or differences. In general, to arrive at an untied ranking that is consistent, it is sufficient to know the outcomes in the ranking of two properly chosen pairs. For this reason, the following theorem will be of value in evaluating required probabilities.

Theorem 2.6.1.

If the joint distribution of

$$Y_{1m} = \frac{y_{1m} - \mu_{1m}}{\sigma_{1m}}, \quad Y_{2n} = \frac{y_{2n} - \mu_{2n}}{\sigma_{2n}}, \quad Y_{3p} = \frac{y_{3p} - \mu_{3p}}{\sigma_{3p}}$$

converges to that of Z_1, Z_2 and Z_3, and if

$$\frac{\sigma_{2n}}{\sigma_{1m}} \to \lambda_1 \quad \text{and} \quad \frac{\sigma_{3p}}{\sigma_{2n}} \to \lambda_2 \quad \text{as} \quad m,\ n,\ p \to \infty,$$

then for real numbers ρ_1 and ρ_2,

$$P\left[\frac{(y_{1m}-\rho_1 y_{2n}) - (\mu_{1m}-\rho_1\mu_{2n})}{(\sigma_{1m}^2 + \rho_1^2\,\sigma_{2n}^2)^{1/2}} \le t_1,\ \frac{(y_{2n}-\rho_2 y_{3p}) - (\mu_{2n}-\rho_2\mu_{3p})}{(\sigma_{2n}^2 + \rho_2^2\,\sigma_{3p}^2)^{1/2}} \le t_2\right]$$

$$\to P\left[Z_1 - \rho_1 Z_2 \le t_1(1+\rho_1^2\,\lambda_1^2)^{1/2},\ \ Z_2 - \rho_2 Z_3 \le t_2(1+\rho_2^2\,\lambda_2^2)^{1/2}\right].$$

Proof of theorem 2.6.1.

Write

$$\frac{(y_{1m}-\rho_1 y_{2n}) - (\mu_{1m}-\rho_1\mu_{2n})}{(\sigma_{1m}^2 + \rho_1^2\,\sigma_{2n}^2)^{1/2}} = \frac{Y_{1m}}{\left(1+\rho_1^2\,\dfrac{\sigma_{2n}^2}{\sigma_{1m}^2}\right)^{1/2}} - \frac{Y_{2n}}{\left(1+\dfrac{\sigma_{1m}^2}{\rho_1^2\sigma_{2n}^2}\right)^{1/2}}$$

and

$$\frac{(y_{2n}-\rho_2 y_{3p}) - (\mu_{2n}-\rho_2\mu_{3p})}{(\sigma_{2n}^2 + \rho_2^2\,\sigma_{3p}^2)^{1/2}} = \frac{Y_{2n}}{\left(1+\rho_2^2\,\dfrac{\sigma_{3p}^2}{\sigma_{2n}^2}\right)^{1/2}} - \frac{Y_{3p}}{\left(1+\dfrac{\sigma_{2n}^2}{\rho_2^2\sigma_{3p}^2}\right)^{1/2}}.$$

If $\varphi_{mnp}(t_1,t_2,t_3)$ is the joint characteristic function of Y_{1m}, Y_{2n}, Y_{3p} and $\varphi(t_1,t_2,t_3)$ is the joint characteristic function of Z_1, Z_2 and Z_3, then $\varphi_{mnp}(t_1,t_2,t_3)$ converges uniformly to $\varphi(t_1,t_2,t_3)$. The joint characteristic function of the random variables

$$\frac{Y_{1m}}{\left(1+\rho_1^2\frac{\sigma_{2n}^2}{\sigma_{1m}^2}\right)^{1/2}} - \frac{Y_{2n}}{\left(1+\frac{\sigma_{1m}^2}{\rho_1^2\sigma_{2n}^2}\right)^{1/2}}, \quad \frac{Y_{2n}}{\left(1+\rho_2^2\frac{\sigma_{3p}^2}{\sigma_{2n}^2}\right)^{1/2}} - \frac{Y_{3p}}{\left(1+\frac{\sigma_{2n}^2}{\rho_2^2\sigma_{3p}^2}\right)^{1/2}}$$

is given as

$$\varphi_{mnp}\left(t_1\left(1+\rho_1^2\frac{\sigma_{2n}^2}{\sigma_{1m}^2}\right)^{1/2}, \quad t_2\left(1+\rho_2^2\frac{\sigma_{3p}^2}{\sigma_{2n}^2}\right)^{1/2} - t_1\left(1+\frac{\sigma_{1m}^2}{\rho_1^2\sigma_{2n}^2}\right)^{1/2}, \right.$$

$$\left. - t_2\left(1+\frac{\sigma_{2n}^2}{\rho_2^2\sigma_{3m}^2}\right)^{1/2} \right), \qquad (2.6.4)$$

which converges to

$$\varphi\left(t_1(1+\rho_1^2\lambda_1^2)^{-1/2}, \quad t_2(1+\rho_2^2\lambda_2^2)^{-1/2} - t_1(1+\rho_1^{-2}\lambda_1^{-2})^{-1/2}, \right.$$

$$\left. - t_2(1+\rho_2^{-2}\lambda_2^{-2})^{-1/2}\right). \qquad (2.6.5)$$

But (2.6.5) is the joint characteristic function of the random variables

$$\frac{Z_1 - \rho_1\lambda_1 Z_2}{(1+\rho_1^2\lambda_1^2)^{1/2}}, \quad \frac{Z_2 - \rho_2\lambda_2 Z_3}{(1+\rho_2^2\lambda_2^2)^{1/2}}.$$

Hence, upon appealing to the uniqueness theorem, the proof is complete.

We now proceed to investigate the three types of distributions, beginning with the Cauchy type.

2.7. The Asymptotic Derivation of the Generalized Bradley-Terry Model for Triple Comparisons

The Cauchy Type Distribution

For a given stimulus, X_i, we shall assume that the experienced sensation, y_{im}, is the maximum of a set of m independent, identically distributed random variables with distribution function, F_i, where F_i (i=1,2,3) is a Cauchy type distribution. The Cauchy type distribution is considered first because the sequences $\{\mu_{im}\}$ (i=1,2,3) can be taken to be sequences of zeros, and the limit distribution,

$$\Psi_2(y;\alpha) = \begin{cases} 0 & y \leq 0 \\ \exp - (-y^{-\alpha}) & y > 0 \end{cases} \qquad \alpha > 0 ,$$

has the property

$$\Psi_2(cy;\alpha) = [\Psi_2(y;\alpha)]^{c^{-\alpha}} \qquad \text{for } c \text{ real.} \tag{2.7.1}$$

Let us suppose that a subject, when exposed to three stimuli, say X_1, X_2 and X_3, compares the experienced sensations, y_{1m}, y_{2n}, y_{3p}, two at a time by considering the ratios y_{1m}/y_{2n}, y_{1m}/y_{3p} and y_{2n}/y_{3p}.

We postulate the existence of a threshold parameter, η, whereby the subject assigns the pairwise rankings in the following way.

$$\begin{aligned} &X_i \rightarrow X_j && \text{if} \quad \log (y_{im}/y_{jn}) > \eta , \\ &X_i = X_j && \text{if} \quad | \log (y_{im}/y_{jn}) | < \eta \\ \text{and} \quad &X_j \rightarrow X_i && \text{if} \quad \log (y_{im}/y_{jn}) < -\eta . \end{aligned} \tag{2.7.2}$$

The subject's total ranking, then, will depend upon the outcome of the three pairwise comparisons.

The probability of a consistent untied ranking, say $P[X_1 \rightarrow X_2 \rightarrow X_3]$, is given by

$$P[X_1 \rightarrow X_2 \rightarrow X_3] = P[y_{1m}/y_{2n} > e^{\eta},\ y_{1m}/y_{3p} > e^{\eta},\ y_{2n}/y_{3p} > e^{\eta}]$$

$$= P[y_{1m}/y_{2n} > e^{\eta},\ y_{2n}/y_{3p} > e^{\eta}]$$

$$= P[y_{2n}/y_{1m} < e^{-\eta},\ y_{3p}/y_{2n} < e^{-\eta}] \ . \qquad (2.7.3)$$

If we assume that the conditions of theorem 2.6.1 are fulfilled with $\sigma_{2n}/\sigma_{1m} \rightarrow \lambda_1$, $\sigma_{3p}/\sigma_{2n} \rightarrow \lambda_2$ and Z_i has the Cauchy limit distribution given in (2.7.1), then (2.7.3) becomes

$$P[X_1 \rightarrow X_2 \rightarrow X_3] = P[y_{2n} - \rho\ y_{1m} < 0,\ y_{3p} - \rho\ y_{2n} < 0],$$

where $\rho = e^{-\eta}$.

Upon standardizing, this converges to

$$P[Z_2 - \rho\ \lambda_2^{-1}\ Z_1 < 0,\ Z_3 - \rho\ \lambda_2^{-1}\ Z_2 < 0]$$

$$= \int_0^{\infty} d\Psi_2(y_1;\alpha) \int_0^{\rho\lambda_1^{-1}y_1} d\Psi_2(y_2;\alpha) \int_0^{\rho\lambda_2^{-1}y_2} d\Psi_2(y_3;\alpha)$$

$$= \frac{1}{1 + (\rho\lambda_2^{-1})^{-\alpha}} \cdot \frac{1}{1 + (\rho\lambda_1^{-1})^{-\alpha} + (\rho^2\lambda_1^{-1}\lambda_2^{-1})^{-\alpha}} \ . \qquad (2.7.4)$$

For $\rho^{-\alpha} = \theta$, $\lambda_2^{-\alpha} = \pi_3/\pi_2$ and $\lambda_1^{-\alpha} = \pi_2/\pi_1$, we can write (2.7.4) as

$$\frac{\pi_1}{\pi_1 + \theta\pi_2 + \theta^2\pi_3} \cdot \frac{\pi_2}{\pi_2 + \theta\pi_3} \ . \qquad (2.7.5)$$

For m, n and p large, we have $\pi_1 \propto \sigma_{1m}^{\alpha}$, $\pi_2 \propto \sigma_{2n}^{\alpha}$ and $\pi_3 \propto \sigma_{3p}^{\alpha}$. Since, for Cauchy limit distributions, the mode of y_i is

$$Y_i = \sigma_{im}\left(\frac{\alpha}{\alpha+1}\right)^{1/\alpha}, \qquad (i = 1,2,3)$$

we can write (2.7.5) in terms of the true sensations, Y_1, Y_2 and Y_3 as

$$P[X_1 \rightarrow X_2 \rightarrow X_3] \approx \frac{Y_1^{\alpha}}{Y_1^{\alpha}+\theta Y_2^{\alpha}+\theta^2 Y_3^{\alpha}} \cdot \frac{Y_2^{\alpha}}{Y_2^{\alpha}+\theta Y_3^{\alpha}} . \tag{2.7.6}$$

When $\theta = 1$ or equivalently, when $\eta = 0$, (2.7.6) reduces to the model proposed by Bradley and Terry given in (2.6.1).

It is interesting to note that according to this model, no circular triads can occur. This is shown by the following theorem, which is proved in the framework of a Cauchy type distribution.

Theorem 2.7.1.

If $X_1 \rightarrow X_2$ and $X_2 \rightarrow X_3$, then $X_1 \rightarrow X_3$.

Proof of theorem 2.7.1.

$$X_1 \rightarrow X_2 \Rightarrow \frac{y_{2n}}{y_{1m}} < \rho, \quad X_2 \rightarrow X_3 \Rightarrow \frac{y_{3p}}{y_{2n}} < \rho$$

Therefore, since all quantities considered here are positive and $\rho < 1$,

$$\frac{y_{3p}}{y_{1m}} < \rho^2 < \rho \Rightarrow X_1 \rightarrow X_3.$$

However, when a ranking involves a tie, basing the total ranking on just two pairs may lead to another type of inconsistency as demonstrated by the next theorem.

Theorem 2.7.2.

If $X_1 = X_2$ and $X_3 \rightarrow X_1$, then $X_2 = X_3$ or $X_3 \rightarrow X_2$.

Proof of theorem 2.7.2.

$$X_1 = X_2 \Rightarrow \rho < y_{1m}/y_{2n} < \rho^{-1}$$

and

$$X_3 \rightarrow X_1 \Rightarrow y_{1m}/y_{3p} < \rho .$$

These two statements imply that $y_{2n}/y_{3p} < 1 < \rho^{-1}$. But if $y_{2n}/y_{3p} < \rho^{-1}$, then $X_2 = X_3$ or $X_3 \rightarrow X_2$.

Let us now look at the probability of a single tie, say $P[X_3 \rightarrow X_1 = X_2]$. In order to avoid arriving at an inconsistent ranking, let us consider

$$\begin{aligned} P[X_3 \rightarrow X_1 = X_2] &= P[\rho < \frac{y_{1m}}{y_{2n}} < \rho^{-1}, \ \frac{y_{1m}}{y_{3p}} < \rho , \ \frac{y_{2n}}{y_{3p}} < \rho] \\ &= P[\rho < \frac{y_{1m}}{y_{2n}} < 1, \quad \frac{y_{2n}}{y_{3p}} < \rho] \\ &\quad + P[\rho < \frac{y_{2n}}{y_{1m}} < 1, \quad \frac{y_{1m}}{y_{3p}} < \rho] . \end{aligned} \tag{2.7.7}$$

Define the events A and B as

$$\begin{aligned} A: &\quad \rho < \frac{y_{1m}}{y_{2n}} < 1, \quad \frac{y_{2n}}{y_{3m}} < \rho \\ B: &\quad \rho < \frac{y_{2n}}{y_{1m}} < 1, \quad \frac{y_{1m}}{y_{3p}} < \rho . \end{aligned} \tag{2.7.8}$$

After standardizing, P[A] converges to

$$\int_0^\infty d\Psi_2(y_3;\alpha) \int_0^{\rho\lambda_2 y_3} d\Psi_2(y_2;\alpha) \int_{\rho\lambda_1 y_2}^{\lambda_1 y_2} d\Psi_2(y_1;\alpha), \tag{2.7.9}$$

which, upon substituting π_2/π_1 for $\lambda_1^{-\alpha}$, π_3/π_2 for $\lambda_2^{-\alpha}$ and θ for $\rho^{-\alpha}$, reduces to

$$\frac{\pi_1^3\pi_2^2(\theta-1)[\pi_2+2\theta\pi_3]}{(\pi_1+\pi_2)(\pi_1+\theta\pi_2)[\pi_1\pi_2+\theta\pi_3(\pi_1+\pi_2)][\pi_1\pi_2+\theta\pi_3(\pi_1+\theta\pi_2)]} . \tag{2.7.10}$$

A similar expression is found for P[B] by interchanging π_1 and π_2. Hence, we find that

$$\begin{aligned} P[X_3 \to X_2 = X_1] &\approx P[A] + P[B] \\ &= \pi_1^2\pi_2^2(\theta-1)\ f(\pi_1,\pi_2,\pi_3,\theta) , \end{aligned} \tag{2.7.11}$$

where $f_3(\pi_1,\pi_2,\pi_3,\theta)$ is a function symmetric in π_1 and π_2. As in (2.7.6), we can express this probability in terms of the sensation parameters, Y_1, Y_2 and Y_3, as

$$P[X_3 \to X_2 = X_1] \approx (Y_1^\alpha)^2\ (Y_2^\alpha)^2\ (\theta-1)\ f_3(Y_1^\alpha,\ Y_2^\alpha,\ Y_3^\alpha,\ \theta). \tag{2.7.12}$$

When considering the event that the outcome is the triple tie, say $X_1 = X_2 = X_3$, we must consider the region

$$C:\quad \rho < \frac{y_1}{y_2} < \rho^{-1},\quad \rho < \frac{y_1}{y_3} < \rho^{-1},\quad \rho < \frac{y_2}{y_3} < \rho^{-1}. \tag{2.7.13}$$

The region C can be partitioned into two disjoint regions, $\mathcal{P}_1(2,3)$ and $\mathcal{P}_2(2,3)$ where $C = \mathcal{P}(2,3) = \mathcal{P}_1(2,3) \cup \mathcal{P}_2(2,3)$ and $\mathcal{P}_1(2,3)$ and $\mathcal{P}_2(2,3)$ are defined as

$$\wp_1(2,3) = \begin{cases} y_2 < y_3 \\ y_3\,\rho < y_1 < y_2\,\rho^{-1} \\ y_3\,\rho < y_2 < y_3 \end{cases}$$

$$\wp_2(2,3) = \begin{cases} y_3 < y_2 \\ y_2\,\rho < y_1 < y_3\,\rho^{-1} \\ y_3 < y_2 < y_3\,\rho^{-1} \end{cases} \qquad (2.7.14)$$

It is immediately evident that $\wp(2,3) = \wp(3,2)$, i.e., the region C is symmetric in y_2 and y_3. It is easily seen that by using the partition $\wp(1,3)$, that $\wp(1,3) = \wp(3,1)$ and, using the partition $\wp(1,2)$, that $\wp(1,2) = \wp(2,1)$. In fact, it can be shown that

$$C = \wp(1,2) = \wp(2,1) = \wp(1,3) = \wp(3,1) = \wp(2,3) = \wp(3,2);$$

that is, the region C, or more simply the range of integration, is symmetric in y_1, y_2 and y_3. Hence we can write

$$P[X_1 = X_2 = X_3] \to \iiint_{\wp} d\Psi_2(y_1;\alpha)\, d\Psi_2(y_2;\alpha)\, d\Psi_3(y_3;\alpha) \;, \qquad (2.7.15)$$

where $\wp$ is one of the six equivalent partitions of C. Upon integrating and substituting π_1, π_2, π_3 and θ as before, we find that

$$P[X_1 = X_2 = X_3] \approx \pi_1^2 \pi_2^2 \pi_3^2 (\theta-1)^2\, g_3(\pi_1,\pi_2,\pi_3,\theta)\,, \qquad (2.7.16)$$

where $g_3(\pi_1,\pi_2,\pi_3,\theta)$ is a function symmetric in the parameters π_1, π_2 and π_3. In terms of the sensation parameters Y_1, Y_2 and Y_3, we can say that

$$P[X_1 = X_2 = X_3] \approx (Y_1^\alpha)^2\,(Y_2^\alpha)^2\,(Y_3^\alpha)^2\,(\theta-1)^2\,g_3(Y_1^\alpha, Y_2^\alpha, Y_3^\alpha, \theta). \tag{2.7.17}$$

In examining (2.7.11), the probability of a single tie and (2.7.16), the probability of a triple tie, we note that when $\theta = 1$, ($\eta = 0$), then each of these events has probability zero.

The Exponential Type Distribution

As in earlier sections we shall assume that y_{im}, the response to stimulus X_i, is the maximum of a set of m independent, identically distributed random variables with a common exponential type distribution. We postulate the existence of a threshold parameter, η, whereby a subject will report

$$\begin{aligned} X_i &\rightarrow X_j &\quad &\text{if} \quad y_{im} - y_{jn} > \eta\,, \\ X_i &= X_j &\quad &\text{if} \quad |\,y_{im} - y_{jn}\,| < \eta \\ \text{and} \quad X_j &\rightarrow X_i &\quad &\text{if} \quad y_{im} - y_{jn} < -\eta\,. \end{aligned} \tag{2.7.18}$$

For $y_{im} = \max\,(u_1, u_2, \ldots\, u_m)$ $(i=1,2,3)$ and properly chosen sequences, $\{\mu_{im}\}$ and $\{\sigma_{im}\}$,

$$\lim_{m \rightarrow \infty} P\left[\frac{y_i - \mu_{im}}{\sigma_{im}}\right] = \Psi_1(y)$$

for

$$\Psi_1(y) = \exp -(e^{-y}) \qquad -\infty < y < \infty\,.$$

Let us consider the probability of an untied ranking, say $X_1 \rightarrow X_2 \rightarrow X_3$. Now

$$P[X_1 \rightarrow X_2 \rightarrow X_3] = P[y_{1m}-y_{2n}>\eta,\ y_{1m}-y_{3p}>\eta,\ y_{2n}-y_{3p}>\eta]$$

$$= P[y_{1m}-y_{2n}>\eta,\ y_{2n}-y_{3p}>\eta]$$

$$= P[y_{2n}-y_{1m}<-\eta,\ y_{3p}-y_{2n}<-\eta]\ . \qquad (2.7.19)$$

Now if $\sigma_{2n}/\sigma_{1m} \rightarrow \lambda_1$ and $\sigma_{3p}/\sigma_{2n} \rightarrow \lambda_2$, then by theorem 2.7.1 we know that for $\rho_1 = \rho_2 = 1$,

$$P\left[\frac{(y_{2n}-y_{1m}) - (\mu_{2n}-\mu_{1m})}{(\sigma_{2n}^2 + \sigma_{1m}^2)^{1/2}} < t_1,\ \ \frac{(y_{3m}-y_{2n}) - (\mu_{3p}-\mu_{2n})}{(\sigma_{3p}^2 + \sigma_{2n}^2)^{1/2}} < t_2\right]$$

$$\rightarrow P\left[Z_2 - Z_1 < t_1(1+\lambda_1^{-2})^{1/2},\ Z_3 - Z_2 < t_2(1+\lambda_2^{-2})^{1/2}\right],$$

(2.7.20)

which for the exponential type distribution equals

$$\left\{1 + e^{-t_1'}\right\}^{-1} \cdot \left\{1 + e^{-t_2'} + e^{-(t_1'+t_2')}\right\}^{-1} \qquad (2.7.21)$$

for $t_1' = t_1(1 + \lambda_1^{-2})^{1/2}$ and $t_2' = t_2(1 + \lambda_2^{-2})^{1/2}$. For large values of m, n and p with $\lambda_1 = \lambda_2 = 1$, $\sigma_{1m}^2 \approx \sigma_{2n}^2 \approx \sigma_{3p}^2$ and we can write $\sigma_{1m}^2 + \sigma_{2n}^2 \approx \sigma_{2n}^2 + \sigma_{3p}^2 \approx \sigma_d^2$. Hence, (2.7.21) is then approximately equal to

$$\left\{1 + e^{\frac{\eta}{\sigma_d} + \frac{(\mu_{2n}-\mu_{1m})}{\sigma_d}} + e^{\frac{2\eta}{\sigma_d} + \frac{(\mu_{3p}-\mu_{2n})}{\sigma_d}}\right\}^{-1} \cdot \left\{1 + e^{\frac{\eta}{\sigma_d} + \frac{(\mu_{3p}-\mu_{2n})}{\sigma_d}}\right\}^{-1},$$

(2.7.22)

which for

$$\log(\pi_3/\pi_2) = \frac{\mu_{3p}-\mu_{2n}}{\sigma_d}, \quad \log(\pi_2/\pi_1) = \frac{\mu_{2n}-\mu_{1m}}{\sigma_d} \text{ and } \theta = e^{\eta}, \qquad (2.7.23)$$

can be written as

$$P[X_1 \rightarrow X_2 \rightarrow X_3] \approx \frac{\pi_1}{\pi_1+\theta\pi_2+\theta^2\pi_3} \cdot \frac{\pi_2}{\pi_2+\theta\pi_3}. \qquad (2.7.24)$$

For large values of m, n and p, $\log \pi_i \propto \frac{\mu_{im}}{\sigma_d}$, $(i=1,2,3)$ and (2.7.24) can be written in terms of the true sensation parameters Y_1, Y_2 and Y_3, as

$$P[X_1 \rightarrow X_2 \rightarrow X_3] = \frac{e^{\frac{Y_1}{\alpha}}}{e^{\frac{Y_1}{\alpha}}+\theta e^{\frac{Y_2}{\alpha}} + \theta^2 e^{\frac{Y_3}{\alpha}}} \cdot \frac{e^{\frac{Y_2}{\alpha}}}{e^{\frac{Y_2}{\alpha}} + \theta e^{\frac{Y_3}{\alpha}}} \qquad (2.7.25)$$

for $\alpha = \sigma_d$.

In a manner similar to that for the Cauchy type distributions, we find that for the single tied ranking $X_3 \rightarrow X_2 = X_1$,

$$P[X_3 \rightarrow X_2 = X_1] \approx \pi_1^2\pi_2^2(\theta-1)\ f_1(\pi_1,\pi_2,\pi_3,\theta)$$

with $f_1(\pi_1,\pi_2,\pi_3,\theta)$ a symmetric function of π_1 and π_2, while for the triple tie, $X_1 = X_2 = X_3$, we find

$$P[X_1 = X_2 = X_3] \approx \pi_1^2\pi_2^2\pi_3^2(\theta-1)^2\ g_1(\pi_1,\pi_2,\pi_3,\theta)$$

for $g_1(\pi_1,\pi_2,\pi_3,\theta)$ a symmetric function of π_1, π_2 and π_3.

The Limited Type Distribution

For a limited type distribution, F, there exists a constant, ω, satisfying the conditions $F(\omega) = 1$ and $F(\omega-\epsilon) < 1$ for every $\epsilon > 0$. Let us assume that in response to stimulus X_i, experienced sensation, y_{im}, is the maximum of a set of m independent, identically distributed random variables with a common limited type distribution. In the triple comparison of stimuli X_1, X_2 and X_3, let us further assume that the upper limits of sensation, ω_1, ω_2 and ω_3, or their consistent estimates, are available and that the subject bases his decision on the ratios

$$\frac{y_{1m} - \omega_1}{y_{2n} - \omega_2}, \quad \frac{y_{1m} - \omega_1}{y_{3p} - \omega_3} \quad \text{and} \quad \frac{y_{2n} - \omega_2}{y_{3p} - \omega_3},$$

and reports that

$$X_i \rightarrow X_j \quad \text{if} \quad \log\left(\frac{y_{im}-\omega_i}{y_{jn}-\omega_j}\right) < -\eta,$$

$$X_i = X_j \quad \text{if} \quad \left|\log\left(\frac{y_{im}-\omega_i}{y_{jn}-\omega_j}\right)\right| < \eta \qquad (2.7.26)$$

and

$$X_j \rightarrow X_i \quad \text{if} \quad \log\left(\frac{y_{im}-\omega_i}{y_{jn}-\omega_j}\right) > \eta .$$

Consider the probability of an untied ranking, say, $P[X_1 \rightarrow X_2 \rightarrow X_3]$. Then according to (2.7.26),

$$P[X_1 \to X_2 \to X_3] = P\left[\frac{y_{1m}-\omega_1}{y_{2n}-\omega_2} < e^{-\eta},\quad \frac{y_{1m}-\omega_1}{y_{3p}-\omega_3} < e^{-\eta},\quad \frac{y_{2n}-\omega_2}{y_{3p}-\omega_3} < e^{-\eta}\right]$$

$$= P\left[\frac{y_{1m}-\omega_1}{y_{2n}-\omega_2} < e^{-\eta},\quad \frac{y_{2n}-\omega_2}{y_{3p}-\omega_3} < e^{-\eta}\right]$$

$$= P\left[\frac{y_{2n}-\omega_2}{y_{1m}-\omega_1} > e^{\eta},\quad \frac{y_{3p}-\omega_3}{y_{2n}-\omega_2} > e^{\eta}\right]. \tag{2.7.27}$$

Since $y_i - \omega_i$ is a negative quantity for $i = 1,2,3$, we have

$$P[X_1 \to X_2 \to X_3] = P\left[(y_{2n}-\omega_2) - e^{\eta}(y_{1m}-\omega_1) < 0,\ (y_{3p}-\omega_3) - e^{\eta}(y_{2n}-\omega_2) < 0\right]. \tag{2.7.28}$$

For $\rho = e^{\eta}$, upon standardizing (2.7.28) we see that

$$P[X_1 \to X_2 \to X_3] \to P[Z_2 - \rho\,\lambda_1^{-1}\,Z_1 < 0,\quad Z_3 - \rho\,\lambda_2^{-1}\,Z_2 < 0]$$

which, when evaluated using the limit function $\Psi_3(y;\alpha) = \exp -(-y)^{\alpha}$ for $y < 0$, is equal to

$$\frac{1}{1+\rho^{-\alpha}\lambda_1^{-\alpha} + \rho^{-2\alpha}\lambda_1^{-\alpha}\lambda_2^{-\alpha}} \cdot \frac{1}{1+\rho^{-\alpha}\lambda_2^{-\alpha}}. \tag{2.7.29}$$

Upon establishing the correspondence given by $\theta = \rho^{-\alpha}$, $\pi_3/\pi_2 = \lambda_2^{-\alpha}$ and $\pi_2/\pi_1 = \lambda_1^{-\alpha}$, we note that

$$P[X_1 \to X_2 \to X_3] \approx \frac{\pi_1}{\pi_1 + \theta\pi_2 + \theta^2\pi_2} \cdot \frac{\pi_2}{\pi_1 + \theta\pi_2}. \tag{2.7.30}$$

For large values of m, n and p, we may take

$$\lambda_1 \approx \frac{\sigma_{2n}}{\sigma_{1m}} \quad \text{and} \quad \lambda_2 \approx \frac{\sigma_{3p}}{\sigma_{2n}} .$$

Therefore, for large numbers of signals, $\pi_1 \propto \sigma_{1m}^{-\alpha}$, $\pi_2 \propto \sigma_{2n}^{-\alpha}$ and $\pi_3 \propto \sigma_{3p}^{-\alpha}$. In order to show the relationship of (2.7.30) to the sensation parameters Y_1, Y_2 and Y_3, we note that for $\alpha > 1$, the mode of the limit distribution is given as $Y_i = \omega_i + \sigma_{im_i}(\frac{\alpha-1}{\alpha})^{1/\alpha}$. Hence, π_i is proportional to $(\omega_i - Y_i)^{-\alpha}$ and

$$P[X_1 \to X_2 \to X_3] \approx \frac{(\omega_1 - Y_1)^{-\alpha}}{(\omega_1 - Y_1)^{-\alpha} + \theta(\omega_2 - Y_2)^{-\alpha} + \theta^2(\omega_3 - Y_3)^{-\alpha}} \cdot \frac{(\omega_2 - Y_2)^{-\alpha}}{(\omega_2 - Y_2)^{-\alpha} + \theta(\omega_3 - Y_3)^{-\alpha}} . \tag{2.7.31}$$

In a manner similar to that given for the Cauchy type distribution, we find that the probabilities for single and triple ties have the same form as (2.7.11) and (2.7.16) developed for the Cauchy type distributions.

If ω_1, ω_2 and ω_3 are unknown, the results are unchanged if consistent estimators of these parameters are available. Theorem 2.5.1 remains valid for each of the three random variables $(y_{im} - \hat{\omega}_{im})/\sigma_{im}$ $(i=1,2,3)$ and the function $g(\underline{x})$ given in the paper by Mann and Wald [14] can be taken to be

$$g(\hat{z}_1, \hat{z}_2, \hat{z}_3) = \left(\frac{y_{1m} - \hat{\omega}_1}{y_{2n} - \hat{\omega}_2} , \frac{y_{2m} - \hat{\omega}_2}{y_{3p} - \hat{\omega}_3}\right) .$$

Then,

$$\left(\frac{y_{1m} - \hat{\omega}_1}{y_{2n} - \hat{\omega}_2} , \frac{y_{2n} - \hat{\omega}_2}{y_{3p} - \hat{\omega}_3}\right) \overset{D}{\to} (Z_1', Z_2') ,$$

where (Z_1', Z_2') is the limiting random variable defined by

$$\left(\frac{y_{1m} - \omega_1}{y_{2n} - \omega_2}, \frac{y_{2n} - \omega_2}{y_{3p} - \omega_3}\right) \xrightarrow{D} (Z_1', Z_2') .$$

Hence, the limiting distribution is unchanged whether ω_1, ω_2 and ω_3 or their consistent estimators are used.

2.8. Comments Regarding Results

It is interesting to note that using the limit theorem approach and considering the sensation response as the maximum of a large number of signals, we arrive at generalized Bradley-Terry models for both pairwise and triple comparisons. In both cases, when θ, the threshold parameter, takes the value one, the models reduce to the untied models. However, we also note that the generalized triple comparison model arrived at using limit theorems is not a practical model in application for the following reasons. First, the forms of the probabilities for single and triple ties are too unwieldy to be easily handled. Secondly, this model lacks symmetry and reversibility. Reversibility is used here in the sense defined by Pendergrass [19]. A model is said to be reversible if the form (and hence the value) of the probabilities is unaltered if the least desirable item is assigned rank one and the most desirable item assigned rank three, rather than using the standard procedure of using rank one to designate the best and rank three to designate the worst item. Thirdly, these probabilities are not independent of the order in which the ranks are assigned. Lastly, since inconsistent rankings can occur when ties are present, the probabilistic model would necessarily have to be conditioned on the event that no inconsistent rankings occur, which would make the form of the probabilities even more unwieldy.

For these reasons, we turn our attention to the problem of developing a tie model for triple comparisons that overcomes most of these difficulties.

CHAPTER 3

GENERALIZATION OF THE BRADLEY-PENDERGRASS MODEL

3.1. Introduction

In the investigation of rank analysis of triple comparisons, models have been postulated as generalizations of pairwise comparison models. Two such models specify that the t treatments to be compared have true ratings, $\pi_1, \pi_2, \ldots, \pi_t$, on a subjective continuum such that $\pi_i \geq 0$ $(i = 1,2,\ldots,t)$ and $\sum_{i=1}^{n} \pi_i = 1$, the latter condition imposed to ensure determinacy of the ratings. A terse examination of these two models and their properties will be beneficial in considering any generalization of a triple comparison model to account for the occurrence of ties in the rankings.

Model I

Bradley and Terry [4] proposed a model wherein the probability that treatment i obtains top ranking in the triple comparison of treatments i, j and k is $\pi_i/(\pi_i + \pi_j + \pi_k)$ and, when this has occurred, the probability that treatment j outranks treatment k is $\pi_j/(\pi_j + \pi_k)$. Hence,

$$P[i \rightarrow j \rightarrow k] = \frac{\pi_i}{(\pi_i + \pi_j + \pi_k)} \cdot \frac{\pi_j}{(\pi_j + \pi_k)} \ . \tag{3.1.1}$$

This model has the desirable property that the unconditional probability that treatment i outranks treatment j is $\pi_i/(\pi_i + \pi_j)$, the same probability as in paired comparisons. However, this model does not have the property of being reversible in the sense that the probability given in (3.1.1) does not retain its form when the direction of ranking is reversed.

Model II

A second model has been proposed by Bradley and Pendergrass [20] whereby the three treatments (i,j,k) are tacitly compared two at a time until a consistent ranking of the three treatments is attained. This model is given as

$$P[i \to j \to k] = \frac{\pi_i^2 \pi_j}{\pi_i^2(\pi_j+\pi_k) + \pi_j^2(\pi_i+\pi_k) + \pi_k^2(\pi_i+\pi_j)} . \qquad (3.1.2)$$

This model has the property that

$$P[i \to j \mid i,j \to k] = P[i \to j \mid k \to i,j] = \pi_i/(\pi_i + \pi_j),$$

the same probability as in paired comparisons. In addition to having the property of reversibility, the model is also unaffected by the order of assignment of the ranks. This last property is important since in some experiments, it may be easier to choose the "worst" item or treatment first, while in others it may be easier to choose the "best" first.

Pendergrass [19] reports that both models yield similar results in applications and, as such, leave little to choose between them.

However, mathematically, model II is to be desired because of its reversibility and symmetry. We therefore chose to generalize this model to account for ties that occur in such ranking experiments.

Before proceeding with any generalization, we note that for the Bradley-Pendergrass model, $P[i \rightarrow j \rightarrow k]$ is actually a conditional probability, conditional upon the event C, that no inconsistent ranking of the three treatments occurs. This is easily seen if we consider the triple comparison of items i, j and k as consisting of the three independent pairwise comparisons involving i and j, i and k, and j and k. In this light,

$$P[i \rightarrow j \rightarrow k] = P[i \rightarrow j] \cdot P[i \rightarrow k] \cdot P[j \rightarrow k]$$

$$= \frac{\pi_i}{\pi_i+\pi_j} \cdot \frac{\pi_i}{\pi_i+\pi_k} \cdot \frac{\pi_j}{\pi_j+\pi_k}$$

$$= \frac{\pi_i^2 \pi_j}{\pi_i^2(\pi_j+\pi_k) + \pi_j^2 \pi_i+\pi_k) + \pi_k^2(\pi_i+\pi_j) - 2\pi_i\pi_j\pi_k} . \qquad (3.1.3)$$

An inconsistent ranking would occur if either of two circular rankings, called triads, occur. The first triad would result when in the three pairwise comparisons, $(i \rightarrow j)$, $(j \rightarrow k)$ and $(k \rightarrow i)$ which gives the total ranking as $(i \rightarrow j \rightarrow k \rightarrow i)$. The second triad would result when $(j \rightarrow i)$, $(k \rightarrow j)$ and $(i \rightarrow k)$ resulting in the circular ranking $(k \rightarrow j \rightarrow i \rightarrow k)$. The probability that a given triad occurs is

$$\frac{\pi_i \pi_j \pi_k}{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k)} .$$

Hence, the probability that no triads occur is given as

$$1 - \frac{2\pi_i\pi_j\pi_k}{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k)} = \frac{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k) - 2\pi_i\pi_j\pi_k}{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k)} . \qquad (3.1.4)$$

Using (3..1..3) and (3.1.4), we see that

$$P[i \rightarrow j \rightarrow k \mid C] = \frac{\pi_i^2\pi_j}{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k)} \cdot \frac{(\pi_i+\pi_j)(\pi_i+\pi_k)(\pi_j+\pi_k)}{\pi_i^2(\pi_j+\pi_k) + \pi_j^2(\pi_i+\pi_k) + \pi_k^2(\pi_i+\pi_j)}$$

$$= \frac{\pi_i^2\pi_j}{\pi_i^2(\pi_j+\pi_k) + \pi_j^2(\pi_i+\pi_k) + \pi_k^2(\pi_i+\pi_j)} , \qquad (3.1.5)$$

which is the probability given by the Bradley-Pendergrass model in (3.1.1).

3.2. The Development of the Generalized Bradley-Pendergrass Model for Triple Comparisons

As we have seen, the Bradley-Pendergrass triple comparison probability model is actually a conditional probability based on three independent pairwise comparisons, using the Bradley-Terry model for each pairwise comparison. In the generalization of the Bradley-Pendergrass model to include tied rankings, we shall also use a conditional probability based on three independent pairwise comparisons, using the probabilities of Rao and Kupper for the outcomes associated with each paired comparison (see equations in (2.2.6)).

With the introduction of a tie parameter, θ, into the model, we should keep the following criteria in mind. (See [7].)

1. The tie model should reduce to the untied model when $\theta = 1$.
2. The probability of a triple tie should tend to one as θ, the tie parameter, tends to infinity, while the probability of any other ranking should tend to zero.
3. The tie model should have the property that

$$\begin{aligned} &P[j \rightarrow k \mid j,k \rightarrow i] = \\ &P[j \rightarrow k \mid i \rightarrow j,k] = \pi_j/(\pi_j + \theta\pi_k) \end{aligned} \tag{3.2.1}$$

4. When $\theta = 1$ and $\pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$, each untied ranking should have probability one-sixth, while for $\pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$, the probability within each of the three types of outcomes (no ties, one tie, three ties) should be the same.
5. The complete set of probabilities should have a common functional form for all outcomes.

To develop a generalized Bradley-Pendergrass model with a tie parameter as a conditional probability, given no inconsistent ranking, we must consider twenty-seven outcomes based on three pairwise comparisons, since each pairwise comparison has three possible outcomes. Of these twenty-seven distinct outcomes,

1. six result in consistent, untied rankings,
2. six result in consistent rankings with one tie,
3. one results in a consistent triple tie and
4. fourteen result in inconsistent rankings.

The following notation is introduced to facilitate the development of a triple comparison tie model:

	Symbol	Definition
1.	D_{ij}	$\pi_i + \theta\pi_j$,
2.	D_{ijk}	$(\pi_i+\theta\pi_j)(\pi_j+\theta\pi_i)(\pi_i+\theta\pi_k)(\pi_k+\theta\pi_i)(\pi_j+\theta\pi_k)(\pi_k+\theta\pi_j)$,
3.	Q_{ijk}	Q_{ijk}/D_{ijk} is the sum of the probabilities associated with the fourteen inconsistencies and
3.	A_{ijk}	$D_{ijk} - Q_{ijk}$. (3.2.2)

After some manipulation Q_{ijk} and A_{ijk} can be written as

$$\begin{aligned} Q_{ijk} = {} & \left\{\pi_i\pi_j\pi_k\,[D_{ji}D_{ik}D_{kj} + D_{ij}D_{jk}D_{ki}]\right\} \\ & + (\theta^2-1)\left\{\pi_i^2\pi_j\pi_k\,[D_{ji}D_{kj} + D_{ki}D_{jk}] + \pi_i\pi_j^2\pi_k\,[D_{kj}D_{ik} + D_{ij}D_{ki}]\right. \\ & \qquad \left. + \pi_i\pi_j\pi_k^2\,[D_{jk}D_{ij} + D_{ik}D_{ji}]\right\} \\ & + (\theta^2-1)^2\left\{\pi_i^2\pi_j^2\pi_k\,[D_{ki} + D_{kj}] + \pi_i^2\pi_j\pi_k^2\,[D_{ij} + D_{ik}]\right. \\ & \qquad \left. + \pi_i\pi_j^2\pi_k^2\,[D_{ji} + D_{jk}]\right\} , \end{aligned}$$

and

$$\begin{aligned} A_{ijk} = {} & \pi_i^2\pi_j\,D_{ji}D_{ki}D_{kj} + \pi_i^2\pi_k\,D_{ji}D_{ki}D_{jk} + \pi_i\pi_k^2\,D_{ji}D_{ik}D_{jk} \\ & + \pi_i\pi_j^2\,D_{ij}D_{ki}D_{kj} + \pi_j^2\pi_k\,D_{ij}D_{ik}D_{kj} + \pi_j\pi_k^2\,D_{ij}D_{ik}D_{jk} \\ & + (\theta^2-1)\left\{\pi_i^2\pi_j\pi_k\,D_{ji}D_{kj} + \pi_i\pi_j^2\pi_k\,D_{ij}D_{kj} + \pi_i\pi_j\pi_k^2\,D_{ik}D_{jk}\right. \\ & \qquad \left. + \pi_i^2\pi_j^2\,D_{ki}D_{kj} + \pi_i^2\pi_k^2\,D_{ji}D_{jk} + \pi_j^2\pi_k^2\,D_{ij}D_{ik}\right\} \\ & + (\theta^2-1)^3\left\{\pi_i^2\pi_j^2\pi_k^2\right\} . \end{aligned} \tag{3.2.3}$$

Having displayed these quantities, we note that, for the event C, the occurrence of no inconsistencies, P[C] is given as

$$P[C] = 1 - \frac{Q_{ijk}}{D_{ijk}} = \frac{A_{ijk}}{D_{ijk}} . \tag{3.2.4}$$

Conditional on C, we can now write the probabilities associated with the three types of consistent outcomes. As examples, consider the following outcomes for the triple (i,j,k).

1. No tie:

$$P[i \to j \to k \mid C] = \frac{\pi_i^2 \pi_j D_{ji} D_{ki} D_{kj}}{A_{ijk}} . \tag{3.2.5}$$

2. One tie:

$$\begin{aligned} &(a)\ P[i = j \to k \mid C] = \frac{\pi_i^2 \pi_j^2 D_{ki} D_{kj} (\theta^2 - 1)}{A_{ijk}} , \\ &(b)\ P[i \to j = k \mid C] = \frac{\pi_i^2 \pi_j \pi_k D_{ji} D_{ki} (\theta^2 - 1)}{A_{ijk}} . \end{aligned} \tag{3.2.6}$$

3. Triple tie:

$$P[i = j = k \mid C] = \frac{\pi_i^2 \pi_j^2 \pi_k^2 (\theta^2 - 1)^3}{A_{ijk}} . \tag{3.2.7}$$

That these probabilities exhibit the five criteria listed earlier is easily shown.

1. By examining A_{ijk} in the form given by $D_{ijk} - Q_{ijk}$, for $\theta = 1$, we see that $D_{ij} = D_{ji}$ and hence

$$A_{ijk} = D_{ij} D_{ik} D_{jk} [D_{ij} D_{ik} D_{jk} - 2\pi_i \pi_j \pi_k]$$

Therefore, when $\theta = 1$, the probability of an untied consistent ranking is given, for example, as

$$P[i \to j \to k \mid C] = \frac{\pi_i^2 \pi_j}{D_{ij} D_{ik} D_{jk} - 2\pi_i \pi_j \pi_k} ,$$

the Bradley-Pendergrass model. For $\theta = 1$, all probabilities involving a tie are identically zero, since these quantities contain the factor $\theta^2 - 1$.

2. Noting that $A_{ijk} = D_{ijk} - Q_{ijk}$ is a polynomial of order 6 in θ and, in particular, D_{ijk} is of order 6 while Q_{ijk} is of order 5, when θ tends to ∞, probabilities of the form (3.2.5) and (3.2.6) tend to zero since the numerators for these quantities exhibit the highest power on θ as the fourth power. To show that the probability of a triple tie tends to one as $\theta \to \infty$, divide both numerator and denominator by θ^6. The numerator tends to $\pi_i^2 \pi_j^2 \pi_k^2$ while in the denominator, $A_{ijk} = D_{ijk} - Q_{ijk}$, D_{ijk}/θ^6 tends to $\pi_i^2 \pi_j^2 \pi_k^2$ and Q_{ijk}/θ^6 tends to zero. Thus the probability of a triple tie tends to one as θ tends to infinity, independently of π_i, π_j and π_k.

3. To demonstrate that $P[j \to k \mid i \to j,k] = \pi_j/(\pi_j + \theta\pi_k)$, we note that

$$P[i \to j,k] = P[i \to j \to k] + P[i \to k \to j] + P[i \to j = k].$$

Then

$$P[j \to k \mid i \to j,k] = \frac{\pi_i^2 \pi_j D_{ji} D_{ki} D_{kj}}{\pi_i^2 \pi_j D_{ji} D_{ki} D_{kj} + \pi_i^2 \pi_k D_{ki} D_{ji} D_{jk} + \pi_i^2 \pi_j \pi_k D_{ji} D_{ki} (\theta^2 - 1)}$$

$$= \frac{\pi_j D_{kj}}{\pi_j D_{kj} + \pi_k D_{jk} + \pi_j \pi_k (\theta^2 - 1)}$$

$$= \frac{\pi_j D_{kj}}{D_{jk} D_{kj}} = \frac{\pi_j}{\pi_j + \theta\pi_k} .$$

In a similar manner, we can show that

$$P[j \to k \mid j,k \to i] = \pi_j/\pi_j+\theta\pi_k.$$

4. If $\theta = 1$ and $\pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$, only (3.2.5) is non-zero with probability one-sixth.

If $\theta > 1$ and $\pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$, then

(a) the probability of a non-tied ranking is $(1+\theta)^3/\Theta$,

(b) the probability of one tie is $(1+\theta)^2(\theta^2-1)/\Theta$, and

(c) the probability of a triple tie is $(\theta^2-1)^3/\Theta$,

where Θ is A_{ijk} evaluated at $\pi_i = \pi_j = \pi_k = \frac{1}{t}$. Hence, the probabilities are the same for outcomes within a given type.

5. To show that all probabilities have a common functional form, we define the following parameters. For the r^{th} replication of triple (i,j,k) $(i \neq j \neq k;\ 1 \leq i,j,k \leq t)$, write

$$a_{ij(k)r} = \begin{cases} 1 \text{ if } i \to j \text{ in the } r^{th} \text{ replicate of } (i,j,k) \\ 0 \text{ otherwise.} \end{cases}$$

$$b_{ij(k)r} = \begin{cases} 1 \text{ if } i = j \text{ in the } r^{th} \text{ replicate of } (i,j,k) \\ 0 \text{ otherwise.} \end{cases}$$

$$c_{ij(k)r} = a_{ij(k)r} + b_{ij(k)r}. \qquad (3.2.8)$$

For the present we assume that there is only one replicate of each triple and drop the subscript r for simplicity.

If (m,n,p) is a permutation of the triple (i,j,k), we can write the probability of a consistent outcome involving the triple (i,j,k) as

$$\frac{\left[\prod_{m}\left(\pi_m^{\sum_{n\neq p} c_{mn(p)}}\right)\right]\cdot\left[\prod_{m\neq n} D_{mn}^{\sum_{p} a_{nm(p)}}\right](\theta^2-1)^{\frac{1}{2}\sum_{m\neq n\neq p} b_{mn(p)}}}{A_{ijk}}\,, \qquad (3.2.9)$$

where the products and sums are over the six permutations of (i,j,k). Therefore, (3.2.9) exhibits a common functional form for the probabilities associated with each of the possible outcomes for the given triple (i,j,k).

In addition to the properties listed, this generalized model is reversible. Suppose rank one is assigned to the "best" item and rank three to the "worst"; then as a specific example

$$P[i \rightarrow j \rightarrow k] = \pi_i^2 \pi_j \, D_{ji} D_{ki} D_{kj} / A_{ijk} \,. \qquad (3.2.10)$$

However, if rank one is assigned to the "worst" item and rank three to the "best," then the equivalent ranking for (3.2.10) would be $k \rightarrow j \rightarrow i$ and

$$P[k \rightarrow j \rightarrow i] = \beta_k^2 \beta_j \, G_{jk} G_{ik} G_{ij} / C_{ijk} \,, \qquad (3.2.11)$$

where $G_{jk} = \beta_j + \theta\beta_k$, $C_{ijk} = A_{ijk}(\beta_i, \beta_j, \beta_k)$ with β_i, β_j and β_k satisfying

$$\frac{\beta_i}{\beta_j} = \frac{\pi_j}{\pi_i}\,, \quad \frac{\beta_i}{\beta_k} = \frac{\pi_k}{\pi_i} \quad \text{and} \quad \frac{\beta_j}{\beta_k} = \frac{\pi_k}{\pi_j}\,. \qquad (3.2.12)$$

Hence, if the π_i exist and are non-zero, then the β_i exist as parameters and have the same relationship to rankings in the reverse direction as do the π_i to the original ranking.

Since the tie model proposed in (3.2.5), (3.2.6) and (3.2.7) or, equivalently, (3.2.9) has the property of conditional independence of the ranking of a pair of items when the third item has been ranked either first or last, the question of the possibility of complete independence of the rank of the third item in the triple arises quite naturally.

With $\theta = 1$, the proposed tie model reduces to the Bradley-Pendergrass model with six possible untied rankings of the items in the triple (i,j,k). In this simpler case, the condition of complete independence requires that the model satisfy the condition,

$$P[i \rightarrow j \mid r_k = 1] = P[i \rightarrow j \mid r_k = 2] = P[i \rightarrow j \mid r_k = 3] , \qquad (3.2.13)$$

where r_k is the rank of item k. Pendergrass [19] has shown that this condition is satisfied only when the six probabilities corresponding to the six possible rankings are all equal. Hence, there can be no general untied model satisfying (3.2.13). Since a generalized model allowing for ties must reduce to the untied model, the untied model is just a special case of a tie model for certain values of the "tie" parameter. Hence, it follows that there can be no general tie model satisfying (3.2.13).

CHAPTER 4

ESTIMATION AND TESTS USING THE GENERALIZED BRADLEY-PENDERGRASS MODEL

4.1. Maximum Likelihood Estimation

In this section we will consider a method for obtaining maximum likelihood estimators for the parameters of the modified triple comparison model.

Let t represent the number of treatments or items under consideration in an experiment. A complete repetition is defined to be the set of all possible distinct triples that can be formed with the t treatments. When $t \geq 3$, there are $\binom{t}{3}$ triples in one complete repetition. The number of complete repetitions in the experiment is denoted by n.

The variables $a_{ij(k)r}$, $b_{ij(k)r}$ and $c_{ij(k)r}$ defined in (3.2.8) are random variables depending upon the ranks of treatments i, j and k in the r^{th} repetition of the triple (i,j,k), ($i \neq j \neq k$; $1 \leq i,j,k \leq t$; $r = 1,2,\ldots,n$).

In order to avoid confusion resulting from the use of the dot notation to indicate the summation over designated subscripts, we shall agree to use the following notation when possible.

Write

(i) $a_{ij(k)} = \sum_r a_{ij(k)r}$; this is the number of times i is preferred to j in the presence of k.

(ii) $a_{ij} = \sum_k a_{ij(k)}$; this is the number of times i is preferred to j in the experiment.

(iii) $a_i = \sum_j a_{ij}$; this is the number of times i is preferred to some other treatment in the experiment.

(iv) $a = \sum_i a_i$; this is the number of times a preference is stated in the experiment.

(v) $b_{ij(k)} = \sum_r b_{ij(k)r}$; this is the number of times i is tied with j in the presence of k.

(vi) $b_{ij} = \sum_k b_{ij(k)}$; this is the number of times that i is tied with j in the experiment.

(vii) $b_i = \sum_j b_{ij}$; this is the number of times that i is tied with some other treatment in the experiment.

(viii) $b = \sum_i b_i$; this is twice the number of ties in the experiment.

It follows from (3.2.9) that

(ix) $c_{ij(k)} = \sum_r c_{ij(k)r}$ is the number of times that i is preferred to or tied with j in the presence of k,

(x) $c_{ij} = \sum_k c_{ij(k)}$ is the number of times that i is preferred to or tied with j in the experiment,

(xi) $c_i = \sum_j c_{ij}$ is the number of times i is preferred to or tied with another treatment in the experiment, and finally,

(xii) $c = \sum_i c_i$ is the number of preferences plus twice the number of ties in the experiment.

Using the parameters just defined, we can write the likelihood function for n replicates of the $\binom{t}{3}$ treatment combinations as proportional to

$$L(\pi_1,\ldots,\pi_{t,}\,\theta) = \frac{\left[\prod_{i=1}^{t} \pi_i^{c_i}\right]\left[\prod_{i\neq j}^{t} D_{ji}^{a_{ij}}\right](\theta^2-1)^{\frac{1}{2}b}}{\prod_{i<j<k} [A_{ijk}]^n} \quad , \qquad 4.1.1)$$

while $\ln L(\pi_1,\ldots,\pi_{t,}\,\theta)$ is proportional to

$$\sum_{i=1}^{t} c_i \ln \pi_i + \frac{b}{2}\ln(\theta^2-1) + \sum_{i\neq j}^{t} a_{ij} \ln D_{ji}$$

$$- n \sum_{i<j<k}^{t} \ln A_{ijk} \quad . \qquad (4.1.2)$$

Maximizing the logarithm of the likelihood function under the restriction that $\sum_{i=1}^{t} \pi_i = 1$, we find that the maximum likelihood estimators, p_i $(i = 1,\ldots,t)$ and $\hat{\theta}$, of π_i $(i = 1,\ldots,t)$ and θ, are the solutions of the equations

$$\frac{c_i}{p_i} + \sum_{j(\neq i)}^{t} \frac{a_{ji}}{d_{ij}} + \hat{\theta} \sum_{j(\neq i)}^{t} \frac{a_{ij}}{d_{ji}} = n \sum_{j<k}^{t} \frac{A_{ijk}^{(i)}}{A_{ijk}} \; , \qquad (i=1,2,\ldots,t) \qquad (4.1.3)$$

$$b\,\frac{\hat{\theta}}{\hat{\theta}^2-1} + \sum_{i\neq j}^{t} \frac{a_{ji}p_j}{d_{ij}} = n \sum_{i<j<k}^{t} \frac{A_{ijk}^{(\theta)}}{A_{ijk}} \qquad (4.1.4)$$

and $$\sum_{i=1}^{t} p_i = 1 \; , \qquad (4.1.5)$$

where $A_{ijk}^{(i)} = \partial A_{ijk}/\partial \pi_i$, $A_{ijk}^{(\theta)} = \partial A_{ijk}/\partial \theta$, and $d_{ij} = p_i + \hat{\theta}p_j$.

To solve this system of equations, it is convenient to write (4.1.3) and (4.1.4) in the form

$$k\, p_i = c_i \left[n \sum_{j<k}^{t} \frac{A^{(i)}_{ijk}}{A_{ijk}} - \sum_{j(\neq i)}^{t} \frac{a_{ji}}{d_{ij}} - \hat{\theta} \sum_{j(\neq i)}^{t} \frac{a_{ij}}{d_{ji}} \right]^{-1} \tag{4.1.6}$$

and

$$\hat{\theta} = 1 + b \frac{\hat{\theta}}{1+\hat{\theta}} \left[n \sum_{i<j<k}^{t} \frac{A^{(\theta)}_{ijk}}{A_{ijk}} - \sum_{i\neq j}^{t} \frac{a_{ji}\, p_j}{d_{ij}} \right]^{-1}, \tag{4.1.7}$$

where the constant k is eliminated through the use of (4.1.5). Explicit solutions of equations (4.1.6) and (4.1.7) are not possible, but iterative solutions are. Starting with a set of trial values, $\hat{\theta}^{(0)}$, $p_i^{(0)}$ $(i = 1,2,\ldots,t)$, an iterative procedure employing (4.1.5), (4.1.6) and (4.1.7) will yield a sequence of estimates, $\hat{\theta}^{(j)}$, $p_i^{(j)}$ $(i = 1,2,\ldots,t)$. The procedure is continued until the process converges to the accuracy required.

A good choice of starting values will shorten the number of iterations required. As an aid in finding starting values for the iteration process, we have considered approximating the likelihood equations by two equations wherein $A^{(i)}_{ijk}$, $A^{(\theta)}_{ijk}$ and A_{ijk} are replaced by the terms involving the highest power of $\pi_i \pi_j \pi_k$. Thus

$$\frac{A^{(i)}_{ijk}}{A_{ijk}} \approx \frac{2}{\pi_i}$$

and (4.1.8)

$$\frac{A^{(\theta)}_{ijk}}{A_{ijk}} \approx \frac{6}{\theta} .$$

Using the maximum likelihood equations with $A^{(i)}_{ijk}/A_{ijk}$ replaced by $2/p_i$ and $A^{(\theta)}_{ijk}/A_{ijk}$ replaced by $6/\hat{\theta}$, we have the following approximate likelihood equations:

$$p_i = \frac{\left[2n \binom{t-1}{2} - c_i\right]}{\left[\sum_{j(\neq i)}^{t} \frac{a_{ji}}{d_{ij}} + \hat{\theta} \sum_{j(\neq i)}^{t} \frac{a_{ij}}{d_{ji}}\right]} \qquad (i=1,2,\ldots,t) \tag{4.1.9}$$

and

$$\hat{\theta} = 1 + b \frac{\hat{\theta}}{(1+\hat{\theta})} \left[\frac{6n}{\hat{\theta}} \binom{t}{3} - \sum_{i \neq j}^{t} \frac{a_{ji}\, p_j}{d_{ij}}\right]^{-1}. \tag{4.1.10}$$

In equation (4.1.10), let $p_i = p \quad (i = 1,2,\ldots,t)$; (4.1.10) then simplifies to

$$\hat{\theta} = 1 + b \frac{\hat{\theta}}{1+\hat{\theta}} \left[\frac{6n}{\hat{\theta}} \binom{t}{3} - \frac{a}{1+\hat{\theta}}\right]^{-1}. \tag{4.1.11}$$

Let us define the constant N by

$$N = 3n \binom{t}{3}, \tag{4.1.12}$$

the number of pairwise comparisons made in the total experiment and therefore

$$N = a + \frac{1}{2} b, \tag{4.1.13}$$

that is, N, the total number of pairwise comparisons, must equal the total number of preferences plus the number of ties. Therefore, employing (4.1.12) and (4.1.13), we can write

$$\hat{\theta} = 1 + b \frac{\hat{\theta}}{1+\hat{\theta}} \left[\frac{2a+b}{\hat{\theta}} - \frac{a}{1+\hat{\theta}}\right]^{-1}, \tag{4.1.14}$$

which simplifies to the quadratic equation,

$$a\, \hat{\theta}^2 + a\, \hat{\theta} - (2a+b) = 0 \tag{4.1.15}$$

with solutions

$$\hat{\theta} = -\frac{1}{2} \pm \frac{1}{2a}(9a^2 + 4ab)^{1/2} . \qquad (4.1.16)$$

Considering only the positive root, we have

$$\hat{\theta} = \sqrt{\frac{9}{4} + \frac{b}{a}} - \frac{1}{2} \geq 1 . \qquad (4.1.17)$$

Hence, we have an admissible value of $\hat{\theta}$ as the solution of (4.1.10) for $p_i = p$ $(i = 1,2,\ldots,t)$.

Using the value of $\hat{\theta}$ found in (4.1.17) as $\hat{\theta}^{(0)}$, we proceed to find an initial value for p_i $(i = 1,2,\ldots,t)$ using equations (4.1.9). Assume all the p_j's except p_i are equal. This implies that $p_j = \frac{1-p_i}{t-1}$ $(j \neq i,\ i = 1,2,\ldots,t)$. Then (4.1.9) simplifies to the quadratic equations in p_i given as

$$p_i^2[a_{i.}\ \{\hat{\theta}(t-1)^2 - \hat{\theta}^2(t-1)\} + a_{.i}\ \{\hat{\theta}^2(t-1) - \hat{\theta}\}]$$

$$+\ p_i[a_{i.}\ \{\hat{\theta}^2(t-1)\} - a_{.i}\ \{\hat{\theta}^2(t-1) - 2\hat{\theta}\}]$$

$$-\ [a_{.i}\ \hat{\theta}] = 0,\ (i = 1,\ldots,t), \qquad (4.1.18)$$

where $a_{i.} = a_i$, while $a_{.i} = \sum_{j(\neq i)}^{t} a_{ji}$, the number of times another treatment was preferred to treatment i. If $p_i = 0$, the value of (4.1.18) is $-a_{.i}\ \hat{\theta} \leq 0$, while for $p_i = 1$, the value of the equation is

$$\hat{\theta}\ [a_{.i} + a_{i.}(t-1)^2] \geq 0.$$

In fact, for $p_i = 0$, (4.1.18) is zero iff $a_{.i}$ is zero and for $p_i = 1$, (4.1.18) is zero iff both $a_{.i}$ and $a_{i.}$ are zero. Therefore, since for either $a_{.i} \neq 0$ or $a_{i.} \neq 0$, (4.1.18) has different signs at $p_i = 0$ and $p_i = 1$, it follows that (4.1.18) has two real unequal roots, exactly one of which lies in the interval (0,1). We can use this root as the initial value, $p_i^{(0)}$ $(i = 1,2,\ldots,t)$, to be used with $\hat{\theta}^{(0)}$ found in (4.1.17), to begin the iteration procedure.

4.2. Tests of Hypotheses

In this section we will consider large-sample tests of three hypotheses which are of practical interest.

Test 1. A test of the equality of the true treatment ratings. We are interested in testing

$$H_{01} : \pi_i = \frac{1}{t} \qquad (i = 1,2,\ldots,t)$$

against the alternative

$$H_{11} : \pi_i \neq \pi_j \quad \text{for some} \quad i \neq j,\ 1 \leq i,\ j \leq t.$$

A large sample test based on the likelihood ratio

$$\Lambda_1 = \frac{\max\limits_{\theta} L(\frac{1}{t}, \frac{1}{t}, \ldots, \frac{1}{t}, \theta)}{L(p_1, p_2, \ldots, p_t, \hat{\theta})}$$

is obtained by using $-2 \ln \Lambda_1$ as a test statistic. Under H_{01}, $-2\ln \Lambda_1$ has an approximate χ^2-distribution with (t-1) degrees of freedom.

Under H_{01}, from (4.1.1) we have

$$L(\tfrac{1}{t}, \tfrac{1}{t}, \ldots, \tfrac{1}{t}, \hat{\theta}_o) = \frac{(\frac{1}{t})^{c} \, (\hat{\theta}_o^2 - 1)^{b/2} \, [\frac{1}{t} (1+\hat{\theta}_o)]^{a}}{[(\frac{1}{t})^6 (1+\hat{\theta}_o)^3 (\hat{\theta}_o^3 - 3\hat{\theta}_o^2 + a\hat{\theta}_o - 1)]^{n\binom{t}{3}}}, \tag{4.2.1}$$

while under H_{11} we have

$$L(p_1, p_2, \ldots, p_t, \hat{\theta}) = \frac{\prod_{i=1}^{t} p_i^{c_i} \, (\hat{\theta}^2 - 1)^{b/2} \prod_{i \neq j}^{t} d_{ij}^{a_{ji}}}{\prod_{i<j<k}^{t} (\hat{A}_{ijk})^{n}} . \tag{4.2.2}$$

By using the relationships $6\binom{t}{3} = 2a + b = a + c$, algebraic simplification shows that for $\chi_1^2 = -2 \ln \Lambda_1$, we have

$$\chi_1^2 = -2 \left\{ \sum_{i=1}^{t} c_i \ln \left(\frac{1}{tp_i}\right) + \frac{b}{2} \ln \left(\frac{\hat{\theta}_o^2 - 1}{\hat{\theta}^2 - 1}\right) \right.$$

$$+ \sum_{i \neq 1}^{t} a_{ji} \ln \left(\frac{1+\hat{\theta}_o}{td_{ij}}\right)$$

$$\left. - n \sum_{i<j<k} \ln \left[\frac{(1+\hat{\theta}_o)^2 (\hat{\theta}_o^3 - 3\hat{\theta}_o^2 + 9\hat{\theta}_o - 1)}{t^6 \hat{A}_{ijk}}\right] \right\}, \tag{4.2.3}$$

where $\hat{\theta}_o$ is the solution to (4.1.7) for $p_i = \frac{1}{t}$ $(i = 1, \ldots, t)$, while $p_1, p_2, \ldots, p_t, \hat{\theta}$ are the solutions to (4.1.6) and (4.1.7) and $\hat{A}_{ijk}$ is A_{ijk} evaluated using $p_1, p_2, \ldots, p_t, \hat{\theta}$.

Test 2. A test of the threshold parameter, θ. Here we are interested in testing whether

$$H_{02} : \theta = \theta_o$$

against the alternative

$$H_{12} : \theta \neq \theta_o .$$

A large sample test of H_{02} is provided by the statistic $-2 \ln \Lambda_2 = \chi_2^2$, where Λ_2 is defined by

$$\Lambda_2 = \frac{\max\limits_{\pi_i} L(\pi_1,\ldots,\pi_t, \theta_o)}{L(p_1,\ldots,p_t, \hat{\theta})} .$$

Under H_{02}, χ_2^2 has an approximate χ^2 distribution with one degree of freedom. If $p_1', p_2',\ldots,p_t'$ are the solutions to (4.1.5) and (4.1.6) with $\theta = \theta_o$, then

$$\chi_2^2 = 2\left\{\sum_{i=1}^{t} c_i \ln\left(\frac{p_i}{p_i'}\right) + \frac{1}{2} b \ln\left(\frac{\hat{\theta}^2-1}{\theta_o^2-1}\right) + \sum_{i\neq j}^{t} a_{ij} \ln\left(\frac{d_{ji}}{d_{ji}'}\right) - n \sum_{i<j<k}^{t} \ln\left(\frac{A_{ijk}}{A_{ijk}'}\right)\right\} , \tag{4.2.4}$$

where d_{ij}' and A_{ijk}' are evaluated using $p_1', p_2',\ldots,p_t'$ θ_o.

An hypothesis of interest would be $\theta = 1$, i.e., the Bradley-Pendergrass model is adequate for the data. However, the appearance of one or more ties in the data is sufficient evidence to reject the Bradley-Pendergrass model, since the observed ties would represent the occurrence of impossible events under the hypothesis that $\theta = 1$.

Test 3. A test of the adequacy of the model. To test the adequacy of the model, a large sample goodness of fit test can be employed using the χ^2 statistic. For each distinct triple, let us define the following thirteen cell probabilities corresponding to the thirteen possible outcomes as

$$P(i \to j \to k), \quad P(i \to k \to j), \ldots, P(i = j = k).$$

Let $p(i \to j \to k)$, $p(i \to k \to j), \ldots, p(i = j = k)$ be the sample estimates found using p_1, $p_2, \ldots, p_t, \hat{\theta}$, the solutions to (4.1.5), (4.1.6) and (4.1.7). For n replicates of the $\binom{t}{3}$ triples, the estimated cell frequencies would be given as

$$\begin{aligned}
f_o(i \to j \to k) &= n\, p(i \to j \to k) \\
f_o(i \to k \to j) &= n\, p(i \to k \to j) \\
&\vdots \\
f_o(i = j = k) &= n\, p(i = j = k)\ .
\end{aligned}$$

The test statistic would be

$$\chi_3^2 = \Sigma \left\{ \frac{[f(i \to j \to k) - f_o(i \to j \to k]^2}{f_o(i \to j \to k)} + \ldots + \frac{[f(i = j = k) - f_o(i = j = k)]^2}{f_o(i = j = k)} \right\}, \tag{4.2.5}$$

where the sum is taken over the $\binom{t}{3}$ treatment combinations. χ_3^2 is approximately distributed as a χ^2-variate with $t[2t^2 - 6t + 3]$ degrees of freedom.

4.3. Combination of Experiments

In certain cases, repetitions of triple comparisons may be performed by different judges in subjective evaluations at different times, in different places or under different conditions. In such situations the overall experiment may be thought of as consisting of g groups of repetitions, the u^{th} group having n_u repetitions, with $\sum_{u=1}^{g} n_u = n$. The parameters for the u^{th} group can be taken to be $(\pi_1, \ldots, \pi_t, \theta_u)$ for $u = 1, 2, \ldots, g$. This assumes that the merit ratings for the t treatments remain constant over the g groups and allows the threshold parameter to change from judge to judge, time to time, or place to place. Since the model is multiplicative with respect to the parameter θ_u, the model does allow for an interaction of treatment ratings and groups.

For g groups of repetitions of the $\binom{t}{3}$ triple comparisons, the likelihood of the observed sample is given as

$$L(\pi_1, \pi_2, \ldots, \pi_t, \theta_1, \theta_2, \ldots, \theta_u)$$

$$= \prod_{u=1}^{g} L(\pi_1, \pi_2, \ldots, \pi_t, \theta_u)$$

$$= \prod_{u=1}^{g} \left\{ \frac{\left[\prod_{i=1}^{t} \pi_i^{c_i^u}\right] \left[(\theta_u^2 - 1)^{\frac{b^u}{2}}\right] \left[\prod_{i \neq j}^{t} {D_{ji}^{u}}^{a_{ij}^{u}}\right]}{\prod_{i<j<k}^{t} [{}^{u}A_{ijk}]^{n_u}} \right\}$$

$$= \frac{\left[\prod_{i=1}^{t} \pi_i^{\sum_{u=1}^{g} c_i^u}\right] \left[\prod_{u=1}^{g} (\theta_u^2 - 1)^{b^u/2}\right] \left[\prod_{u=1}^{g} \prod_{i \neq j}^{t} {D_{ji}^{u}}^{a_{ij}^{u}}\right]}{\prod_{u=1}^{g} \prod_{i<j<k}^{t} [{}^{u}A_{ijk}]^{n_u}} \; .$$

The maximum likelihood estimators are the iterative solutions to the equations

$$kp_i = \sum_{u=1}^{g} c_i^u \left[\sum_{u=1}^{g} \sum_{j<k}^{t} n_u \frac{{}^uA_{ijk}^{(i)}}{{}^uA_{ijk}} - \sum_{u=1}^{g} \sum_{j(\neq i)}^{t} \theta_u \frac{a_{ij}^u}{D_{ji}^u} - \sum_{u=1}^{g} \sum_{j(\neq i)}^{t} \frac{a_{ji}^u}{D_{ij}^u} \right]^{-1}$$

and $i = 1,2,\ldots,t$ (4.3.1)

$$\hat{\theta}_u = 1 + b^u \frac{\hat{\theta}_u}{1+\hat{\theta}_u} \left[n_u \sum_{i<j<k}^{t} \frac{{}^uA_{ijk}^{(\theta)}}{{}^uA_{ijk}} - \sum_{i\neq j}^{t} \frac{a_{ij}^u \, p_i}{D_{ji}^u} \right]^{-1} \qquad (4.3.2)$$

$u = 1,2,\ldots,g$,

where the constant k is eliminated through the use of the equation

$$\sum_{i=1}^{t} p_i = 1 \ . \qquad (4.3.3)$$

As in (4.1.9) and (4.1.10), we can get starting values for the iterative procedure required to solve (4.3.1) and (4.3.2) using (4.3.3) by considering approximate likelihood equations given as

$$kp_i = \frac{\left[2n \binom{t-1}{2} - \sum_{u=1}^{g} c_i^u \right]}{\sum_{u=1}^{g} \sum_{j(\neq i)}^{t} \left[\frac{a_{ji}^u}{d_{ij}^u} + \hat{\theta}_u \frac{a_{ij}^u}{d_{ji}^u} \right]} \qquad i = 1,2,\ldots,t \qquad (4.3.4)$$

and

$$\hat{\theta}_u = 1 + b^u \frac{\hat{\theta}_u}{(1+\hat{\theta}_u)} \left[\frac{6n_u \binom{t}{3}}{\hat{\theta}_u} - \sum_{i\neq j}^{t} \frac{a_{ji}^u \, p_j}{d_{ij}} \right]^{-1} \qquad u = 1,2,\ldots,g , \qquad (4.3.5)$$

again eliminating k through the use of

$$\sum_{i=1}^{t} p_i = 1 \ .$$

In equation (4.3.5), let $p_i = p$ $(i=1,2,\ldots,t)$. Then (4.3.5) simplifies to the quadratic equation,

$$a^u \hat{\theta}_u^2 + a^u \hat{\theta}_u - (2a^u + b^u) = 0 , \qquad (4.3.6)$$

which has an admissible solution ≥ 1. Using the values of $\hat{\theta}_u$ $(u=1,2,\ldots,g)$ found as solutions to the g equations in (4.3.6), g estimates of p_i can be found for each of the g groups, by solving the g equations

$$(p_i^u)^2 [a_i^u \{\hat{\theta}_u (t-1)^2 - \hat{\theta}_u^2 (t-1)\} + a_{.i} \{\hat{\theta}_u^2 (t-1) - \hat{\theta}_u\}]$$

$$+ p_i^u [a_{i.}^u \{\hat{\theta}_u^2 (t-1)\} - a_{.i} \{\hat{\theta}_u^2 (t-1) - 2\hat{\theta}\}] - [a_{.i}^u \hat{\theta}] = 0 \qquad (4.3.7)$$

$$u = 1,2,\ldots,g .$$

Therefore, one can use as the initial value of p_i, the weighted average of the p_i^u's, given as

$$p_i = \sum_{u=1}^{g} n_u p_i^u / n , \qquad (i = 1,2,\ldots,t) \qquad (4.3.8)$$

which will yield admissible solutions, since each equation in (4.3.7) yields an admissible solution based on the data for the u^{th} group.

Two tests involving this model are of importance and will be considered here.

Test 4. Test of equality of the treatment ratings. Consider

$$H_{04} : \pi_i = \frac{1}{t} \qquad (i=1,2,\ldots,t)$$

versus the alternative

$$H_{14} : \pi_i \neq \pi_j \qquad (i \neq j;\ 1 \leq i,j \leq t).$$

Based on the likelihood ratio

$$\Lambda_4 = \frac{\max\limits_{(\theta_1,\ldots,\theta_u)} L(\frac{1}{t},\ldots,\frac{1}{t}, \theta_1,\ldots,\theta_u)}{L(p''_1, p''_2,\ldots,p''_t, \hat{\theta}_1,\ldots,\hat{\theta}_u)}$$

where $p''_1, p''_2,\ldots,p''_t, \hat{\theta}_1,\ldots,\hat{\theta}_u$ are the iterative solutions to (4.3.1), (4.3.2) and (4.3.3), $\chi_4^2 = -2 \ln \Lambda_4$ has an approximate χ^2 distribution with (t-1) degrees of freedom. One can show that

$$\chi_4^2 = -2 \sum_{u=1}^{g} \left\{ \sum_{i=1}^{t} c_i^u \ln (tp''_i)^{-1} + \frac{b^u}{2} \ln \left(\frac{\hat{\theta}_{ou}^2 -1}{\hat{\theta}_u^2 -1} \right) \right.$$

$$+ \sum_{i \neq j}^{t} a_{ji}^u \ln \left(\frac{1+\hat{\theta}_{ou}}{td''_{ij}} \right)$$

$$\left. - n \sum_{i<j<k} \ln \left[\frac{(1+\hat{\theta}_{ou})^2 (\hat{\theta}_{ou}^3 -3\hat{\theta}_{ou}^2 +9\hat{\theta}_{ou} -1)}{t^6 \, {}^u A_{ijk}} \right] \right\}, \qquad (4.3.9)$$

where $\hat{\theta}_{ou}$ is the solution to (4.3.2) using the values $\pi_1 = \frac{1}{t},\ldots,\pi_t = \frac{1}{t}$ and ${}^u A_{ijk}$ is evaluated using the values $p''_1,\ldots,p''_t, \hat{\theta}_1,\ldots,\hat{\theta}_g$.

Test 5. A test of equality of the threshold parameters in the different groups. We now test

$$H_{05} : \theta_u = \theta \quad (u = 1,2,\ldots,g) \text{ for some } \theta$$

against the alternative

$$H_{15} : \theta_u \neq \theta_{u'} \quad \text{for some } u \neq u'.$$

The likelihood ratio test will provide the test statistic given as $\chi_5^2 = -2 \ln \Lambda_5$, which is approximately distributed as a χ^2 with (g-1) degrees of freedom for large samples. Now

$$\Lambda_5 = \frac{\max\limits_{\pi_i} L\ (\pi_1,\pi_2,\ldots,\pi_t,\ \theta_o)}{L(p_1'',p_2'',\ldots,p_t'',\ \hat{\theta}_1,\hat{\theta}_2,\ldots,\hat{\theta}_g)} ,$$

where $p_1''\ p_2'',\ldots,p_t'',\ \hat{\theta}_1,\hat{\theta}_2,\ldots,\hat{\theta}_g$ are the solutions of (4.3.1), (4.3.2) and (4.3.3). Denoting the maximizing values of $\pi_1,\ \pi_2,\ldots,\pi_t$ in $L(\pi_1,\pi_2,\ldots,\pi_t,\ \theta_o)$ as $p_1,p_2,\ldots,p_t$, i.e., the solutions to (4.3.1) and (4.3.3) for $\theta_u = \theta_o$, it is easily seen that

$$\chi_5^2 = 2\left[\sum_{i=1}^{t} \sum_{u=1}^{g} c_i^u \ln\left(\frac{p_i''}{p_i}\right) + \sum_{u=1}^{g} b^u \ln\left(\frac{\hat{\theta}_u^2-1}{\theta_o^2-1}\right)\right.$$

$$\left. + \sum_{u=1}^{g} \sum_{i\neq j} a_{ij}^u \ln\left(\frac{d_{ji}''}{d_{ji}^o}\right) - \sum_{u=1}^{g} \sum_{i<j<k}^{t} n_u \frac{{}^u A_{ijk}}{{}^u A_{ijk}^o}\right]. \qquad (4.3.10)$$

We note that all of the preceding methods and results can be adapted to the case of unequal repetitions of the treatment triples with slight modifications, e.g., replacing n with $\sum\limits_{i<j<k} r_{ijk}$ for r_{ijk} the number of repetitions of the treatment triple (i,j,k).

CHAPTER 5

LARGE SAMPLE PROPERTIES OF ESTIMATORS AND TESTS

5.1. The Limiting Distribution of the Maximum Likelihood Estimators

Under general conditions (see for example Wilks [28] or Chanda [5]) which can be verified for our problem, $p_1,\ldots,p_t$, $\hat{\theta}$ are consistent estimators of $\pi_1,\pi_2,\ldots,\pi_t$, θ, and, considering $\pi_t = 1 - \sum_{i=1}^{t} \pi_i$, $\sqrt{n}\,(\hat{\theta}-\theta)$, $\sqrt{n}\,(p_1-\pi_1),\ldots,\sqrt{n}\,(p_{t-1}-\pi_{t-1})$ have an asymptotic t-variate normal distribution with zero means and dispersion matrix $(\lambda^*_{ij})^{-1}$ $(0 \leq i,j \leq t-1)$ where

$$\lambda^*_{ij} = E\left[-\frac{\partial^2 \ln f(\pi,\theta)}{\partial\pi_i\,\partial\pi_j}\right], \quad (1 \leq i,j \leq t-1)$$

$$\lambda^*_{oo} = E\left[-\frac{\partial^2 \ln f(\pi,\theta)}{\partial\,\theta^2}\right], \qquad (5.1.1)$$

$$\lambda^*_{oj} = E\left[-\frac{\partial^2 \ln f(\pi,\theta)}{\partial\theta\,\partial\pi_j}\right] \quad (j = 1,2,\ldots,t-1)$$

and $f(\pi,\theta)$ is the likelihood function for the r^{th} repetition of the experiment. To facilitate finding the elements of (λ^*_{ij}), define

$$\alpha_{ijk} = E(a_{ijk}]$$

$$= \frac{\pi_i^2 \pi_j D_{ji} D_{ki} D_{kj}}{A_{ijk}} + \frac{\pi_i^2 \pi_k D_{ki} D_{ji} D_{jk}}{A_{ijk}} + \frac{\pi_k^2 \pi_i D_{ik} D_{jk} D_{ji}}{A_{ijk}}$$

$$+ \frac{\pi_i^2 \pi_j \pi_k D_{ji} D_{ki} (\theta^2-1)}{A_{ijk}} + \frac{\pi_k^2 \pi_i^2 D_{ji} D_{ki} (\theta^2-1)}{A_{ijk}} ,$$

$$\beta_{ijk} = E[b_{ijk}]$$

$$= \frac{\pi_i^2 \pi_j^2 D_{ki} D_{kj} (\theta^2-1)}{A_{ijk}} + \frac{\pi_k^2 \pi_i \pi_j D_{ik} D_{jk} (\theta^2-1)}{A_{ijk}}$$

$$+ \frac{\pi_i^2 \pi_j^2 \pi_k^2 (\theta^2-1)^3}{A_{ijk}}$$

and

$$\gamma_{ijk} = E[c_{ijk}]$$

$$= E[a_{ijk} + b_{ijk}]$$

$$= \alpha_{ijk} + \beta_{ijk} . \tag{5.1.2}$$

To avoid further confusion with subscripts, let us write the log of the likelihood function for the r^th replicate as

$$\ln f(\pi,\theta) = \frac{b}{2} \ln (\theta^2-1) + \sum_{i=1}^{t} c_i \ln \pi_i + \sum_{i\neq j}^{t} a_{ij} \ln D_{ji}$$

$$- \sum_{i<j<k}^{t} \ln A_{ijk} . \tag{5.1.3}$$

Let D_ν represent the differential operator with respect to π_ν $(\nu = 1,2,\ldots,t)$. Then

$$\lambda_{\nu\nu} = E[-D_\nu^2 \ln f(\pi,\theta)]$$

$$= \frac{\gamma_\nu}{\pi_\nu^2} + \sum_{i(\neq\nu)}^{t} \frac{\alpha_{i\nu}}{D_{\nu i}^2} + \theta^2 \sum_{i(\neq\nu)}^{t} \frac{\alpha_{\nu i}}{D_{i\nu}^2} + \sum_{j<k}^{t} B_{\nu jk}^{(\nu\nu)} \quad (\nu=1,2,\ldots,t), \qquad (5.1.4)$$

where

$$B_{\nu jk}^{(\nu\nu)} = \frac{A_{\nu jk}\, A_{\nu jk}^{(\nu\nu)} - A_{\nu jk}^{(\nu)}\, A_{\nu jk}^{(\nu)}}{[A_{\nu jk}]^2}$$

for

$$A_{\nu jk}^{(\nu)} = D_\nu(A_{\nu jk}) \quad \text{and} \quad A_{\nu jk}^{(\nu\nu)} = D_\nu^2(A_{\nu jk}) \; .$$

For $D_{\nu\mu}$, the differential operator with respect to π_ν and π_μ, we have for the pair (μ,ν) $(\mu\neq\nu;\ 1 \leq \mu,\nu = t)$,

$$\lambda_{\mu\nu} = E[-D_{\mu\nu} \ln f(\pi,\theta)]$$

$$= \theta\left[\frac{\alpha_{\mu\nu}}{D_{\nu\mu}^2} + \frac{\alpha_{\nu\mu}}{D_{\mu\nu}^2}\right] + \sum_{k=1}^{t} B_{\nu\mu k}^{(\nu\mu)} \quad , \qquad (5.1.5)$$

where

$$B_{\nu\mu k}^{(\nu\mu)} = \frac{A_{\nu\mu k}\, A_{\nu\mu k}^{(\nu\mu)} - A_{\nu\mu k}^{(\nu)}\, A_{\nu\mu k}^{(\mu)}}{[A_{\nu\mu k}]^2}$$

for

$$A_{\nu\mu k}^{(\nu\mu)} = D_{\nu\mu}\,(A_{\nu\mu k}).$$

In an analogous manner, we find that

$$\begin{aligned}\lambda_{oo} &= E[- D_{\theta}^{2} \ln f(\pi,\theta)] \\ &= \frac{(\theta^2+1)}{(\theta^2-1)^2}\beta + \sum_{i\neq j}^{t} \frac{\pi_i^2 \alpha_{ij}}{D_{ji}^2} + \sum_{i<j<k}^{t} B_{ijk}^{(\theta,\theta)}\end{aligned} \qquad (5.1.6)$$

and

$$\begin{aligned}\lambda_{o\nu} &= E[- D_{\theta\nu} \ln f(\pi,\theta)] \\ &= \sum_{i(\neq\nu)}^{t} \left[\frac{\pi_i \alpha_{i\nu}}{D_{\nu i}^2} + \theta\, \pi_\nu \frac{\alpha_{\nu i}}{D_{i\nu}^2}\right] + \sum_{j<k}^{t} B_{\nu jk}^{(\nu\theta)} .\end{aligned} \qquad (5.1.7)$$

On differentiating $\ln f(\pi,\theta)$ under the restriction $\pi_t = 1 - \sum_{i=1}^{t} \pi_i$, we find that

$$\begin{aligned}\lambda_{\nu\mu}^{*} &= \lambda_{\nu\mu} - \lambda_{\nu t} - \lambda_{\mu t} + \lambda_{tt} , \\ \lambda_{oo}^{*} &= \lambda_{oo} \\ \text{and} \quad \lambda_{o\nu}^{*} &= \lambda_{o\nu} - \lambda_{ot} \qquad (1 \leq \mu,\nu \leq t-1).\end{aligned} \qquad (5.1.8)$$

It follows that $\sqrt{n}\,(\hat{\theta} - \theta)$, $\sqrt{n}\,(p_1 - \pi_1),\ldots,\sqrt{n}\,(p_t - \pi_t)$ have asymptotically a (t+1)-variate singular normal distribution of rank t with zero means and dispersion matrix $\Sigma = (\sigma_{ij})$, where

$$\Sigma = \left[\begin{array}{c|c} (\lambda_{ij}^{*})^{-1} & \lambda_t^{*} \\ \hline \lambda_t^{*'} & \sigma_{tt} \end{array}\right] \qquad (5.1.9)$$

for

$$\lambda_t^{*} = (\sigma_{o,t}, \sigma_{1,t},\ldots,\sigma_{t-1,t}) ,$$

$$\sigma_{it} = -\sum_{u=1}^{t-1} \sigma_{iu} \qquad (i = 0,1,2,\ldots,t-1)$$

and

$$\sigma_{tt} = \sum_{u=1}^{t-1} \sum_{v=1}^{t-1} \sigma_{uv} .$$

5.2. An Approximation for Σ in the Case of Equal Treatment Ratings

If $p_1, p_2, \ldots, p_t$ are approximately equal, estimates of σ_{ij} may be obtained by taking $\pi_1 = \ldots = \pi_t = \frac{1}{t}$. Let

$$A(\theta) = \frac{3(1+\theta)^3 + 4(1+\theta)^2 (\theta^2-1) + (\theta^2-1)^3}{\Theta} ,$$

$$B(\theta) = \frac{3(1+\theta)^3 + 2(1+\theta)^2 (\theta^2-1)}{\Theta} ,$$

$$C(\theta) = \frac{4\theta(\theta+1)(3\theta^2+3\theta+1)}{\Theta} - 2 ,$$

$$D(\theta) = \frac{2(\theta+1)(\theta^5-\theta^4+\theta^3+11\theta^2+6\theta-1)}{\Theta} - 2 ,$$

$$\Theta = (1+\theta)^3 (\theta^3-3\theta^2+9\theta-1) ,$$

$$\Theta' = t^{-5} A^{(\theta)}_{\nu jk} \Big|_{\pi_i=\frac{1}{t}} ,$$

$$\Theta'' = t^{-4} A^{(\theta\theta)}_{\nu\mu k} \Big|_{\pi_i=\frac{1}{t}}$$

and $$G(\theta) = [\Theta\, \Theta'' - \Theta'^2]/\Theta^2 . \qquad (5.2.1)$$

Now, when $\pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$,

$$\lambda_{\nu\nu} = \lambda_1(\theta,t) \ ,$$

where

$$\lambda_1(\theta,t) = t^2(t-1)(t-2)A(\theta) + t^2(t-1)(t-2)\left[\frac{1+\theta^2}{(1+\theta)^2}\right] B(\theta) + \frac{t^2(t-1)(t-2)}{2} C(\theta), \qquad (5.2.2)$$

and

$$\lambda_{\mu\nu} = \lambda_2(\theta,t),$$

where

$$\lambda_2(\theta,t) = t^2(t-2) \left[\frac{2\theta}{(1+\theta)^2}\right] B(\theta) + t^2(t-2)\ D(\theta) \ . \qquad (5.2.3)$$

Since

$$\lambda^*_{\nu\nu} = \lambda_{\nu\nu} - 2\lambda_{\nu t} + \lambda_{tt} \ ,$$

we can write

$$\lambda^*_{\nu\nu} = 2[\lambda_1(\theta,t) - \lambda_2(\theta,t)] \ . \qquad (5.2.4)$$

In the case $\nu \neq \mu$, we have

$$\lambda^*_{\mu\nu} = \lambda_{\mu\nu} - \lambda_{\mu t} - \lambda_{\nu t} + \lambda_{tt} \ ,$$

which can be written as

$$\lambda^*_{\mu\nu} = [\lambda_1(\theta,t) - \lambda_2(\theta,t)] \ . \qquad (5.2.5)$$

Let us define $\lambda(\theta,t)$ by

$$\lambda(\theta,t) = \lambda_1(\theta,t) - \lambda_2(\theta,t) \ . \qquad (5.2.6)$$

From (5.2.4), (5.2.5) and (5.2.6), we see that

$$\lambda^*_{\nu\nu} = 2\lambda^*_{\mu\nu} = 2\lambda(\theta, t) \tag{5.2.7}$$

for $\mu \neq \nu, \quad 1 \leq \mu, \nu \leq t-1$.

The expression for (5.2.5) can be written as

$$\lambda(\theta,t) = t^2(t-1)(t-2)A(\theta) + t^2(t-1)(t-2)\left[\frac{1+\theta^2}{(1+\theta)^2}\right]B(\theta)$$

$$+ \frac{t^2(t-1)(t-2)}{2}C(\theta) - t^2(t-2)\left[\frac{2\theta}{(1+\theta)^2}\right]B(\theta)$$

$$- t^2(t-2)D(\theta) . \tag{5.2.8}$$

(Expression (5.2.8) will be used in the investigation of power and asymptotic relative efficiency in Sections 5.3 and 5.4.)

We also find that

$$\lambda^*_{oo} = \lambda_{oo}$$

$$= \frac{t(t-1)(t-2)}{3}\left\{\frac{(\theta^2+1)}{(\theta^2-1)^2}A(\theta) + \frac{3}{(1+\theta)^2}B(\theta) + G(\theta)\right\} \tag{5.2.9}$$

and

$$\lambda^*_{o\nu} = \lambda_{o\nu} - \lambda_{ot}$$

$$= 0 \qquad (\nu = 1,2,\ldots,t-1) . \tag{5.2.10}$$

Then the approximated dispersion matrix of the random variables $\sqrt{n}\,(\hat{\theta}-\theta)$, $\sqrt{n}\,(p_1-\pi_1),\ldots,\sqrt{n}\,(p_t-\pi_t)$, when $\pi_1, \pi_2, \ldots, \pi_{t-1}$ and π_t are approximately equal to $1/t$, is given as Σ, whose elements are

$$\sigma_{oo} = [\lambda^*_{oo}]^{-1} ,$$

$$\sigma_{oi} = \sigma_{io} = 0 \qquad (i = 1,2,\ldots,t),$$

$$\sigma_{ii} = \frac{(t-1)}{t}[\lambda(\theta,t)]^{-1} \qquad (i = 1,2,\ldots,t)$$

and

$$\sigma_{ij} = -\frac{1}{t}[\lambda(\theta,t)]^{-1} \qquad (i \neq j;\ 1 \leq i,j \leq t) . \qquad (5.2.11)$$

When $\theta = 1$, the model reduces to that of Bradley and Pendergrass and the value of Σ reduces to the values given by them for $\pi_1 = \ldots = \pi_t = \frac{1}{t}$ as

$$\sigma_{ij} = -3[t^4(t-2)]^{-1} \qquad (i \neq j;\ 1 \leq i,j \leq t)$$

and

$$\sigma_{ii} = 3(t-1)[t^4(t-2)]^{-1} \qquad (i = 1,2,\ldots,t). \qquad (5.2.12)$$

5.3. A Comparison of the Powers of Tests

The investigation of the power of a statistical test serves as a means of assessing the merits of the test in contrast to alternate procedures. Some important alternatives to the method of triple comparisons with ties are (1) the method of triple comparisons without ties, (2) the analysis of variance run in complete blocks and (3) the analysis of variance run in balanced incomplete blocks of size three.

The correspondence between the parameters of the method of paired comparisons and those of analysis of variance have been established by Bradley [1] by referring to the Thurstone-Mosteller model, given as

$$P[X_i \rightarrow X_j] = \frac{1}{\sqrt{2\pi}} \int_{-(S_i-S_j)/\sigma_d}^{\infty} e^{-\frac{1}{2}y^2} dy \ , \tag{5.3.1}$$

where S_i is the location point for the variate X_i on a subjective continuum and σ_d^2 is the variance of the difference (X_i-X_j). Under the analysis of variance assumption that $\rho = 0$, $\sigma_d^2 = 2\sigma^2$, where σ^2 is the common variance of X_i and X_j. Hence, the Thurstone-Mosteller model may be associated with an analysis of variance model having the same variance, σ^2, for the random element of its additive model and with S_i as the effect of treatment i.

In this same paper, Bradley has shown that in comparing (5.3.1) with the Bradley-Terry model for paired comparisons given as

$$P[X_i \rightarrow X_j] = \frac{1}{4a} \int_{-a(\ln \pi_i - \ln \pi_j)}^{\infty} \operatorname{sech}^2 \left(\frac{y}{2a}\right) dy \ , \tag{5.3.2}$$

the form of the integrand does not matter for large sample sizes. In fact, for

$$P[X_i \rightarrow X_j] = \int_c^{\infty} f(y)\, dy \ ,$$

the variance of an estimate, $\hat{c}$, of c obtained from an estimate, p, of $P[X_i \rightarrow X_j]$ does not depend upon the form of f(y), but only upon the ordinate f(-c). Accordingly, Bradley has established the correspondence between S_i and π_i to be

$$S_i = \sqrt{\pi/4} \ln \pi_i \tag{5.3.3}$$

by taking the scale factor in (5.3.2) to be $\sqrt{\pi/8}$ so that $f(0) = 1/\sqrt{2\pi}$ for each model. The value $y = 0$ has been chosen because we wish to consider the test $H_o : \pi_i = \frac{1}{t}$ versus the alternative hypothesis

$$H_1 : \pi_i = \frac{1}{t} + \frac{\delta'_{in'}}{\sqrt{n'}} \qquad (i = 1,2,\ldots,t) \ , \tag{5.3.4}$$

where n' is the number of complete repetitions of the pairwise experiment and $\lim_{n' \to \infty} \delta'_{in'} = \delta'_i$; therefore, $(\ln \pi_i - \ln \pi_j)$ is of order $1/\sqrt{n}$.

Since the correspondence of the parameters of the method of analysis of variance has been established for pairwise comparisons, we need an association of the $\delta'_{in'}$ of pairwise comparisons with the δ_{in} of triple comparisons. To obtain the required association, we note that for paired comparisons with ties

$$P[X_i \rightarrow X_j] = \pi_i/(\pi_i + \theta\pi_j) \ . \tag{5.3.5}$$

For $\pi_i = \frac{1}{t} + \delta'_{in'}/\sqrt{n'}$, (5.3.5) can be expressed as

$$P[X_i \rightarrow X_j] = \frac{1}{1+\theta} + \frac{\theta t}{(1+\theta)^2}\left(\frac{\delta'_{in'} - \delta'_{jn'}}{\sqrt{n'}}\right) + 0\ (\frac{1}{n'}) \ . \tag{5.3.6}$$

From the probability model for triple comparisons with ties, we find

$$\begin{aligned} P[X_i \rightarrow X_j] = {} & P[X_i \rightarrow X_j \rightarrow X_k] + P[X_i \rightarrow X_k \rightarrow X_j] \\ & + P[X_k \rightarrow X_i \rightarrow X_j] + P[X_i \rightarrow X_j = X_k] \\ & + P[X_k = X_i \rightarrow X_j] \ . \end{aligned} \tag{5.3.7}$$

Therefore,

$$P[X_i \rightarrow X_j] = A_{ijk}^{-1}\,[\pi_i^2\,\pi_j\,D_{ji}\,D_{ki}\,D_{kj} + \pi_i^2\,\pi_k\,D_{ki}\,D_{ji}\,D_{jk}$$

$$+ \pi_k^2\,\pi_i\,D_{ik}\,D_{jk}\,D_{ji} + \pi_i^2\,\pi_j\,\pi_k\,D_{ji}D_{ki}(\theta^2-1)$$

$$+ \pi_i^2\,\pi_k^2\,D_{ji}D_{jk}(\theta^2-1)]\ , \tag{5.3.8}$$

which, under the alternate hypothesis,

$$H_1 : \pi_i = \frac{1}{t} + \frac{\delta_{in}}{\sqrt{n}}\ , \tag{5.3.9}$$

where n is the number of complete repetitions of the experiment and $\lim_{n\rightarrow\infty} \delta_{in} = \delta_i$, can be written as

$$P[X_i \rightarrow X_j] = \frac{1 + 2\theta}{6+6(1-\theta)+(1-\theta)^3} + \frac{\theta(1+3\theta)t}{(1+\theta)[6+6(1-\theta)+(1-\theta)^3]} \cdot \frac{(\delta_{in} - \delta_{jn})}{\sqrt{n}}$$

$$+ 0(\frac{1}{n})\ . \tag{5.3.10}$$

It is helpful to think of (5.3.6) and (5.3.10) in the following way. Consider

$$P[X_i \rightarrow X_j \mid \pi = \pi_o + \frac{1}{\sqrt{n}}\,\delta] = P[X_i \rightarrow X_j \mid \pi = \pi_o]$$

$$+ \{P[X_i \rightarrow X_j \mid \pi = \pi_o + \frac{1}{\sqrt{n}}\,\delta]$$

$$- P[X_i \rightarrow X_j \mid \pi = \pi_o]\}\ . \tag{5.3.11}$$

This is to say that the first quantity represents the probability that i is preferred to j under the null hypothesis, while the second quantity represents the contribution due to the fact that the alternate hypothesis is true. In order to establish the correspondence between $\delta'_{in'}$ and δ_{in}, we shall require that the rates of convergence to the null value, π_o, be the same for equal values of n. In other words, we require that

$$\frac{\theta t}{(1+\theta)^2} \frac{(\delta'_{in'}-\delta'_{jn'})}{\sqrt{n'}} = \frac{\theta(1+3\theta)t}{(1+\theta)[6+6(1-\theta)+(1-\theta)^3]} \frac{(\delta_{in}-\delta_{jn})}{\sqrt{n}} , \qquad (5.3.12)$$

which implies that for $n = n'$

$$\delta_{in} = \frac{[6 + 6(1-\theta) + (1-\theta)^3]}{(1+\theta)(1+3\theta)} \delta'_{in}$$

$$= g(\theta)\ \delta'_{in}, \text{ say} . \qquad (5.3.13)$$

We now proceed to investigate the power of the likelihood ratio test, using the triple comparison model with ties as compared to the Bradley-Pendergrass model and the two analysis of variance models.

Under mild regularity conditions, similar to those required for maximum likelihood estimation which hold for our problem, Wald [27] has shown that in testing $H_o : \pi_i = \frac{1}{t}$ versus $H_1 : \pi_i = \frac{1}{t} + \frac{\delta_{in}}{\sqrt{n}}$ $(i = 1,2,\ldots,t)$, the likelihood ratio statistic, $-2 \log \Lambda_1$, given in (4.2.3), is asymptotically distributed as a central χ^2 variate with (t-1) degrees of freedom when H_o is true. When H_1 is true, $-2 \log \Lambda_1$ has an asymptotic χ^2 distribution with (t-1) degrees of freedom and non-centrality parameter given as

$$\varphi = \underline{\delta}' \Sigma^{-1} \underline{\delta} = \lambda(\theta,t) \sum_{i=1}^{t} \delta_i^2 ,$$

for $\lambda(\theta,t)$ defined in (5.2.8). Hence, the asymptotic power for a size α test is given by

$$\beta(\varphi;\alpha) = \int_{\chi^2_{t-1}(\alpha)}^{\infty} f(t;\varphi)\, dt ,$$

where $\chi^2_{t-1}(\alpha)$ is the α-level significance value of a central χ^2 distribution with (t-1) degrees of freedom and $f(t;\varphi)$ is the non-central χ^2 distribution with (t-1) degrees of freedom and non-centrality parameter φ. P. C. Tang [23] has provided tables for evaluating the integral

$$\beta(\varphi;\alpha) = \int_{F_\alpha}^{\infty} f(F';p,q,\varphi)\, dF'$$

for $f(F';p,q,\varphi)$ the density of a non-central F variate with p and q degrees of freedom and non-centrality parameter φ. By taking $q = \infty$, these tables can be used to evaluate (5.3.13).

In investigating the asymptotic power functions of analysis of variance models for (1) a randomized block design with block size t and (2) a balanced incomplete block design with block size three, we note that when σ is known, the test statistic in each case is χ^2 with (t-1) degrees of freedom and non-centrality parameters

$$\varphi_1 = \frac{\pi}{4} \frac{t^2(t-1)(t-2)}{2} \sum_{i=1}^{t} \delta_i^2$$

and

$$\varphi_2 = \frac{\pi}{4} \frac{t^3(t-2)}{3} \sum_{i=1}^{t} \delta_i^2, \qquad (5.3.14)$$

respectively. For triple comparisons with ties, the non-centrality parameter of the asymptotic χ^2 distribution with (t-1) degrees of freedom is given as

$$\varphi_3 = \lambda(\theta,t) \sum_{i=1}^{t} \delta_i'^2 . \tag{5.3.15}$$

Using the correspondence between δ_i and δ_i' established in (5.3.12), we can write (5.3.15) as

$$\varphi_3 = \lambda(\theta,t)\, g^2(\theta) \sum_{i=1}^{t} \delta_i^2 . \tag{5.3.16}$$

Under the analysis of variance models as well as the Bradley-Pendergrass model, the probability of observing one or more ties is zero. This is equivalent to evaluating the generalized Bradley-Pendergrass model at $\theta = 1$, since the probability of one or more ties becomes zero and the model reduces to the untied Bradley-Pendergrass model. Hence, when $\theta = 1$, (5.3.16) gives the non-centrality parameter for the Bradley-Pendergrass model which is

$$\varphi_3 = \frac{t^3(t-2)}{3} \left(\frac{3}{4}\right)^2 \sum_{i=1}^{t} \delta_i^2$$

$$= \frac{3t^3(t-2)}{16} \sum_{i=1}^{t} \delta_i^2 . \tag{5.3.17}$$

To plot the power functions for these tests, we computed $\varphi_i' = (\varphi_i/t)^{1/2}$ in order to use Tang's tables. The power curves for the analysis of variance F tests differed slightly for four treatments and blocks of size three, with the power of the F for complete blocks being higher than that for incomplete blocks of size three. Hence, the asymptotic power functions for the triple comparison test with ties and that for the analysis of variance F test for balanced incomplete blocks of size three have been plotted and are given in Figures 1, 2 and 3. These power functions arise in testing the equality of treatment ratings.

In Figures 1 and 2 we have plotted the power, β, against $\Delta = \sum_{i=1}^{t} \delta_i^2$ for $t = 4$ and 0.05 and 0.01 levels of significance, respectively. The β-scale is logarithmic and the power curves were computed by using Tang's tables. Figures 1 and 2 each exhibit the power curve for the triple comparison test evaluated at $\theta = 1$ in order that this curve can be compared with the F test for balanced incomplete blocks at equal values of Δ. When $\Delta = 0$, $\beta = 0.05$ or 0.01 and, as $\Delta \to \infty$, $\beta \to 1$ for both tests. However, the F test is more powerful than the triple comparison test for all values of Δ.

Recall that in the original discrimination scheme, a judge declared a tie if $|y_i - y_j| < \eta$ for $\log \theta = \eta$. Then values of θ used in Figure 3, namely, $\theta = 1, 3, 4$ and 10, correspond to values of η given as $\eta = 0, 1.10, 1.39$ and 2.30, respectively.

In Figure 3 the power curves for T evaluated at $\theta = 1, 3, 4$ and 10, as well as the curve for F, are given. In plotting β against Δ, it appears that the triple comparison test is more powerful than the analysis of variance F test for the same value of Δ. However, it is important to note that the non-centrality parameter for the F test is given as

$$\varphi_2 = \frac{\pi}{4} \frac{t^3(t-2)}{3} \Delta ,$$

while the non-centrality parameter for the triple comparison test with ties is given as

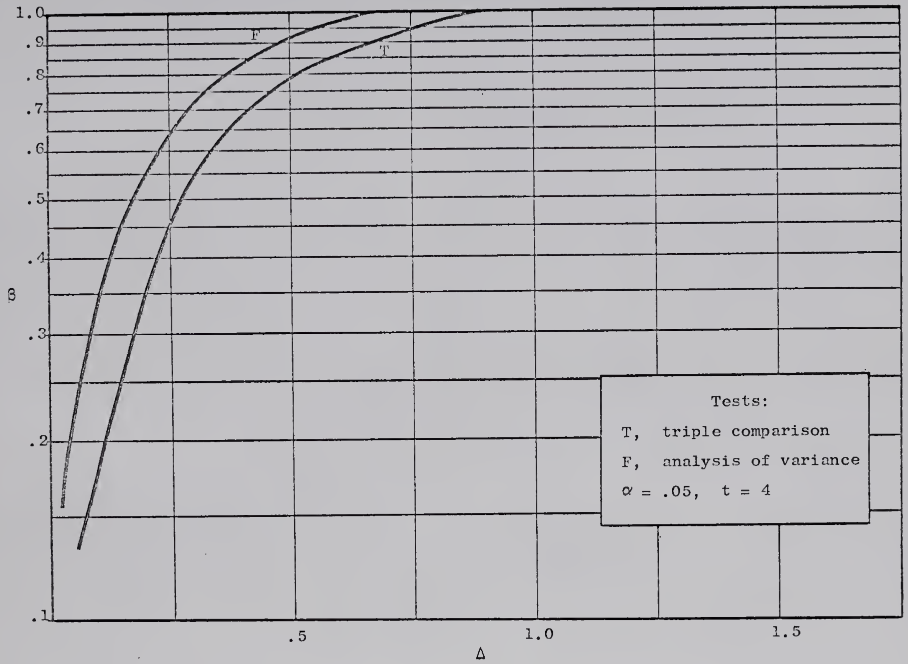

Figure 1. Asymptotic Power Functions for $t = 4$, $\alpha = .05$

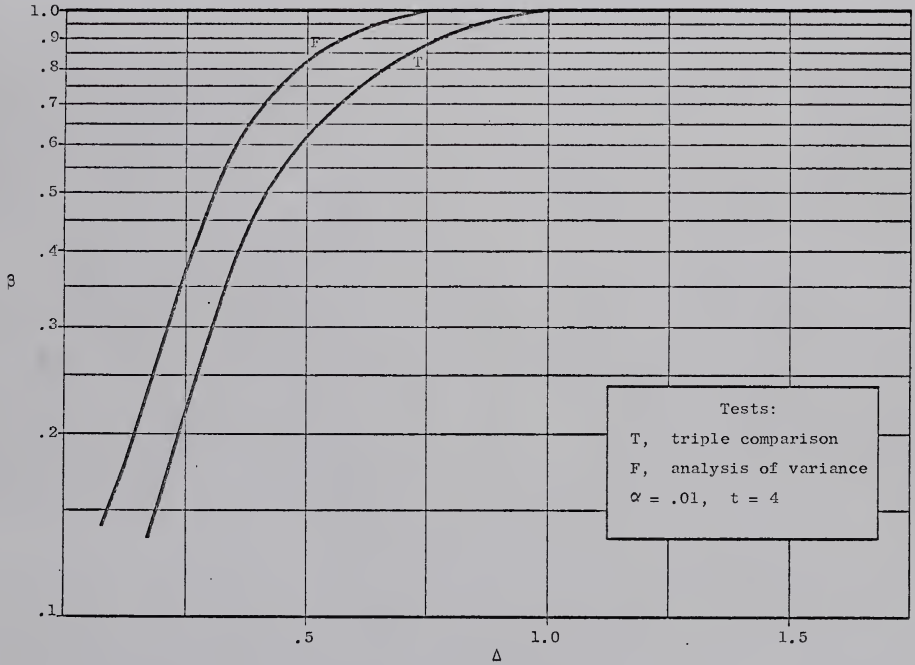

Figure 2. Asymptotic Power Functions for $t = 4$, $\alpha = .01$

$$\varphi_3 = \lambda(\theta,t)\ g^2(\theta)\ \Delta$$

which, when evaluated at $\theta = 1$, becomes

$$\varphi_3\Big|_{\theta=1} = \frac{3t^3(t-2)}{16}\ \Delta\ .$$

By allowing θ to assume values greater than one, we have actually introduced a new value of Δ, say Δ', found by equating φ_3 and $\varphi_3\big|_{\theta=1}$ and given by

$$\Delta' = \frac{16}{3}\ \frac{\lambda(\theta,t)\ g^2(\theta)}{t^3(t-2)}\ \Delta\ . \qquad (5.3.18)$$

In comparing the power of two tests for testing $H_o : \pi_i = \frac{1}{t}$ $(i = 1,\ldots,t)$ versus $H_o : \pi_i = \frac{1}{t} + \frac{\delta_{in}}{\sqrt{n}}$, one does require that the comparison be made for equal alternatives. Therefore, when plotting β against Δ for values of θ greater than one, as in Figure 3, we are comparing the power curves at points along the Δ axis that are not equivalent. However, when β is plotted against Δ', the power curves for $T(\theta)$ obtained using the values $\theta = 3$, 4 and 10 are identical to the power curve given for $T(\theta)$ evaluated at $\theta = 1$.

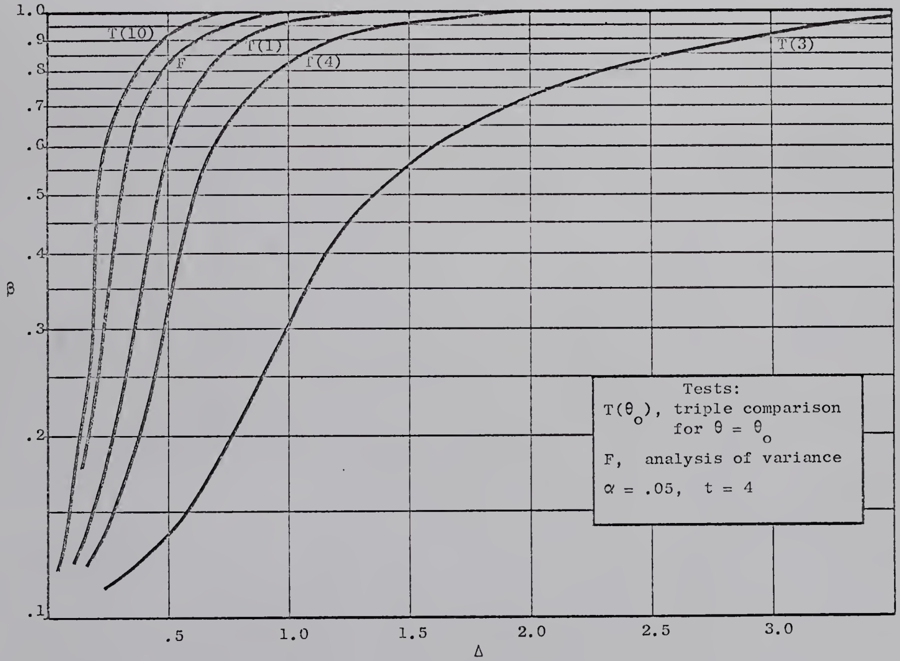

Figure 3. Asymptotic Power Functions for $t = 4$, $\alpha = .01$

5.4. Asymptotic Relative Efficiencies

The theorems of Pitman and Noether [17] yield asymptotic efficiences in terms of the efficacies of the tests being compared. For a statistic T_n, based on a sample of size n, define

$$E[T_n] = \Psi_n\ (\Delta;T_n) \text{ and } \operatorname{Var}[T_n] = \sigma_n^2(\Delta;T_n).$$

The efficacy of the test $H_o : \Delta = \Delta_o$ against the alternative $H_1 : \Delta > \Delta_o$ is

$$R_n^{1/m\delta}\ (\Delta_o;T_n) = \left[\Psi_n^{(m)}(\Delta_o;T_n)/\sigma_n(\Delta_o;T_n)\right]^{1/m\delta}, \qquad (5.4.1)$$

where $\Delta_n = \Delta_o + k/n^\delta$ for $\delta > 0$ and m the order of the last non-zero derivative of $\Psi_n(\Delta;T_n)$ with respect to Δ.

For our problem we have four tests of the hypothesis

$$H_o : \pi_i = \frac{1}{t} \text{ versus } H_1 : \pi_i = \frac{1}{t} + \frac{\delta_{in}}{\sqrt{n}}, \qquad (5.4.2)$$

where

$$\delta_{in} \to \delta_i \text{ as } n \to \infty \qquad (i = 1,2,\ldots,t) .$$

The statistics under comparison will be designated as T_n for the method of triple comparisons with ties, T_n' for the method of triple comparisons with no ties, F_n for the analysis of variance with complete blocks and F_n' for the analysis of variance with balanced incomplete blocks, all of which have asymptotic non-central χ^2-distributions with (t-1) degrees of freedom and non-centrality parameters

$$\varphi_{T_n} = \lim_{n\to\infty} n\,\lambda(\theta,t) \sum_{i=1}^{t} \frac{\delta_i'^2}{n} ,$$

$$\varphi_{T_n'} = \lim_{n\to\infty} n\,\frac{t^3(t-2)}{3} \sum_{i=1}^{t} \frac{\delta_i'^2}{n} ,$$

$$\varphi_{F_n} = \lim_{n\to\infty} n\,\frac{\pi}{4}\,\frac{t^2(t-1)(t-2)}{2} \sum_{i=1}^{t} \frac{\delta_i^2}{n}$$

and

$$\varphi_{F_n'} = \lim_{n\to\infty} n\,\frac{\pi}{4}\,\frac{t^3(t-2)}{3} \sum_{i=1}^{t} \frac{\delta_i^2}{n} . \tag{5.4.3}$$

Taking $\Delta = \sum_{i=1}^{t} \delta_i^2/n$, the null hypothesis is true when $\Delta = \Delta_o = 0$. Corresponding to $\Psi_n(\Delta;T_n)$ in (5.4.1), we may take[1]

$$\Psi_n(\Delta;T_n) = (t-1) + n\,\lambda(\theta,t)\,g^2(\theta)\,\Delta ,$$

$$\Psi_n(\Delta;T_n') = (t-1) + n\,\frac{t^3(t-2)}{3}\left(\frac{3}{4}\right)^2 \Delta ,$$

$$\Psi_n(\Delta;F_n) = (t-1) + n\,\frac{\pi}{4}\,\frac{t^2(t-1)(t-2)}{2}\,\Delta$$

and

$$\Psi_n(\Delta;F_n') = (t-1) + n\,\frac{\pi}{4}\,\frac{t^3(t-2)}{3}\,\Delta . \tag{5.4.4}$$

Differentiation with respect to Δ yields (for all values of Δ, including $\Delta = 0$)

---

[1] The values of Ψ_n and σ_n used here are not the means and variances of the statistics for finite n, but rather were calculated from the limiting distributions. Noether has noted that this is permissible under conditions which can be verified here.

$$\Psi_n'(0;T_n) = n\,\lambda(\theta,t)\,g^2(\theta),$$

$$\Psi_n'(0;T_n') = \frac{3n\,t^2(t-2)}{16},$$

$$\Psi_n'(0;F_n) = n\,\frac{\pi}{4}\,\frac{t^2(t-1)(t-2)}{2}$$

and

$$\Psi_n'(0;F_n') = n\,\frac{\pi}{4}\,\frac{t^3(t-2)}{3}\,. \qquad (5.4.5)$$

From the non-central χ^2 distribution with (t-1) degrees of freedom,

$$\sigma_n^2(0;T_n) = \sigma_n^2(0;T_n') = \sigma_n^2(0;F_n) = \sigma_n^2(0;\sqrt{n'}) = 2(t-1)\,. \qquad (5.4.6)$$

The efficacies of the four tests for $m = 1$ and $\delta = 1$ are

$$R_n(0;T_n) = \frac{n\,\lambda(\theta,t)\,g^2(\theta)}{\sqrt{2(t-1)}},$$

$$R_n(0;T_n') = \frac{3n\,t^3(t-2)/16}{\sqrt{2(t-1)}},$$

$$R_n(0;F_n) = \frac{n\pi\,t^2(t-1)(t-2)/8}{\sqrt{2(t-1)}}$$

and

$$R_n(0;F_n') = \frac{n\pi.t^3(t-2)/12}{\sqrt{2(t-1)}}\,. \qquad (5.4.7)$$

The relative efficiencies of these tests are given by the limits of the ratios of corresponding values of R_n. Hence, the relative efficiencies of interest are given as

$$\text{R.E.}(T \text{ to } T') = \frac{16\lambda(\theta,t)\ g^2(\theta)}{3t^2(t-2)}\ ,$$

$$\text{R.E.}(T \text{ to } F) = \frac{8\lambda(\theta,t)\ g^2(\theta)}{\pi t^2(t-1)(t-2)}$$

and

$$\text{R.E.}(T \text{ to } F') = \frac{12\lambda(\theta,t)\ g^2(\theta)}{\pi t^3(t-2)}\ . \qquad (5.4.8)$$

Table 5.4.1 gives the asymptotic relative efficiencies of the triple comparison test with ties to the triple comparison test without ties, the F test for complete blocks and the F' test for incomplete blocks of size 3 for $t = 3,4,\ldots,10$ and $\theta = 1$, 3, 4 and 10. The asymptotic relative efficiencies for $\theta > 1$ exhibit the same pattern displayed in Figure 3, namely, increasing efficiency for large values of θ. As mentioned in Section 5.3, these comparisons were made for equal values of $\Delta = \sum_{i=1}^{t} \delta_i^2$. Using a rescaling technique similar to that given by (5.3.17), we find that the resulting asymptotic relative efficiencies are those given in Table 5.4.1 for $\theta = 1$.

For testing $H_o : \pi_i = 1/t$ versus $H_1 : \pi_i = 1/t + (\delta_{in})/(\sqrt{n})$ with both analysis of variance and triple comparison techniques, we developed expressions for the probability that treatment i was preferred to j as functions of t, θ, δ_{in} and n for both paired and triple comparisons. In order to relate back to the analysis of variance model, a correspondence between the δ_{in}'s of the paired and triple comparisons was required. We established this correspondence by adjusting the rates of convergence under H_1, $\pi_i = 1/t + (\delta_{in})/(\sqrt{n})$,

TABLE 5.4.1

ASYMPTOTIC RELATIVE EFFICIENCIES

t	3	4	5	6	7	8	9	10
$\theta = 1$								
(T to T')	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
(T to F)	0.72	0.64	0.60	0.57	0.56	0.55	0.54	0.53
(T to F')	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72
$\theta = 3$								
(T to T')	0.28	0.28	0.28	0.28	0.28	0.28	0.28	0.28
(T to F)	0.20	0.18	0.17	0.16	0.16	0.15	0.15	0.15
(T to F')	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
$\theta = 4$								
(T to T')	0.63	0.63	0.63	0.63	0.63	0.63	0.63	0.63
(T to F)	0.45	0.40	0.37	0.36	0.35	0.34	0.34	0.33
(T to F'	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45
$\theta = 5$								
(T to T')	0.92	0.92	0.92	0.92	0.92	0.92	0.92	0.92
(T to F)	0.66	0.59	0.55	0.53	0.51	0.50	0.49	0.49
(T to F')	0.66	0.66	0.66	0.66	0.66	0.66	0.66	0.66
$\theta = 10$								
(T to T')	1.80	1.80	1.80	1.80	1.80	1.80	1.80	1.80
(T to F)	1.29	1.14	1.07	1.03	1.00	0.98	0.97	0.95
(T to F')	1.29	1.29	1.29	1.29	1.29	1.29	1.29	1.29

to the null value given under H_o, $\pi_i = 1/t$, $(i = 1,\ldots,t)$ in such a way that we were at equal distances from the hypothesized value for $P[X_i \rightarrow X_j]$ under H_o. Had a different criterion been used in adjusting these rates of convergence, slightly different power curves and asymptotic relative efficiencies might result. However, when $\theta = 1$, our adjustment factor, $g(\theta)$, and the expressions for $P[X_i \rightarrow X_j]$ are those given by Pendergrass [19].

5.5. A Numerical Example

We now illustrate the use of the proposed generalized Bradley-Pendergrass model using computer-simulated data. The generated data represent data collected for a triple comparison experiment with ties for five treatments, where there are fifty replications of each of ten distinct treatment triples. The values of the treatment ratings were chosen to be $\pi_1 = 0.0666667$, $\pi_2 = 0.1333333$, $\pi_3 = 0.2000000$, $\pi_4 = 0.2666667$ and $\pi_5 = 0.3333333$, while the value of the threshold parameter was chosen as $\theta = 1.5$. The data generated using these values in the model are presented in Table 5.5.1. For each of the ten possible triples, there are thirteen cells. To facilitate the presentation of the cell counts, let us define the following notation. For the triple (i,j,k) where $i < j < k$, the cell numbers one through thirteen correspond to the outcomes:

TABLE 5.5.1

DATA OF NUMERICAL EXAMPLE

	Triples									
Cells	(1,2,3)	(1,2,4)	(1,3,4)	(2,3,4)	(1,2,5)	(1,3,5)	(2,3,5)	(1,4,5)	(2,3,5)	(3,4,5)
1	10	9	12	9	8	11	8	13	11	9
2	13	15	16	8	17	18	11	17	11	6
3	7	6	3	8	7	3	10	2	7	7
4	0	1	0	4	1	1	1	1	1	4
5	2	2	3	1	2	2	1	2	1	4
6	1	2	1	2	1	2	2	2	2	1
7	2	0	2	2	1	0	1	0	1	3
8	3	5	2	5	3	3	6	3	3	6
9	5	3	5	4	3	4	3	5	6	3
10	0	0	0	0	0	0	0	0	0	0
11	2	2	2	2	1	2	2	2	3	2
12	5	5	4	5	6	4	5	3	4	5
13	0	0	0	0	0	0	0	0	0	0
Totals	50	50	50	50	50	50	50	50	50	50

1. $j \rightarrow k \rightarrow i$
2. $k \rightarrow j \rightarrow i$
3. $k \rightarrow i \rightarrow j$
4. $i \rightarrow k \rightarrow j$
5. $j \rightarrow i \rightarrow k$
6. $i \rightarrow j \rightarrow k$
7. $i = j \rightarrow k$
8. $i = k \rightarrow j$
9. $j = k \rightarrow i$
10. $i \rightarrow j = k$
11. $j \rightarrow i = k$
12. $k \rightarrow i = j$
13. $i = j = k$. (5.5.1)

After obtaining a set of initial values for $\hat{\theta}$ and $p_1,\ldots,p_5$, using equations (4.1.15) and (4.1.18), equations (4.1.16) and (4.1.17) were used to obtain the maximum likelihood estimates for the parameters. Seven iterations were required to obtain three-place precision. The initial and final estimates of the parameters as well as the true values used in generating the data are given in Table 5.5.2.

TABLE 5.5.2

MAXIMUM LIKELIHOOD ESTIMATES

	$\hat{\theta}$	p_1	p_2	p_3	p_4	p_5
Initial Values	1.336	0.089	0.163	0.212	0.250	0.285
Final Values	1.478	0.094	0.155	0.204	0.249	0.298
True Values	1.500	0.067	0.133	0.200	0.267	0.333

We have noted in estimating θ, π_1, $\pi_2,\ldots,\pi_5$, using different sets of generated data, that accuracy on the extreme values of the π_i is sacrificed for accuracy on the intermediate values, using the iterative procedure.

In addition to examining the final estimates of θ, $\pi_1,\ldots,\pi_4$ and π_5, it might also be instructive to compare the true cell probabilities with the estimated cell probabilities found using the maximum likelihood solutions given in Table 5.5.2. Table 5.5.3 gives the true and estimated cell probabilities for the triples(1,4,5) and (2,3,4); the first triple involves the three values of p_i $(i = 1,4,5)$ with the largest error in estimation, while the second involves the three values of p_i $(i = 2,3,4)$ with the least error in estimation.

In referring to Table 5.5.3, we find that for the triple (2,3,4) the largest deviation of the estimated from the actual cell probabilities is 0.03510, while for the triple (1,4,5) the largest deviation of the estimated from the actual cell probabilities is 0.05481. These deviations reflect not only the error in estimating $\pi_1,\ldots,\pi_5$, but also the underestimate of θ.

TABLE 5.5.3

ACTUAL AND ESTIMATED CELL PROBABILITIES
FOR THE TRIPLES (1,4,5) AND (2,3,4)

	Triple (2,3,4)		Triple (1,4,5)	
Cells	Actual	Estimated	Actual	Estimated
1	0.15432	0.14409	0.26100	0.23084
2	0.21787	0.18277	0.34108	0.28627
3	0.13407	0.13216	0.06700	0.09113
4	0.05866	0.07543	0.01025	0.02358
5	0.06752	0.08224	0.03992	0.05972
6	0.04155	0.05947	0.00784	0.01901
7	0.02597	0.03313	0.00713	0.01443
8	0.04190	0.04647	0.00985	0.01903
9	0.09078	0.07718	0.14830	0.12237
10	0.02444	0.03186	0.00446	0.01008
11	0.04823	0.05066	0.03838	0.04819
12	0.08380	0.07362	0.06091	0.06918
13	0.01091	0.01093	0.00389	0.00617

A test of $H_{01} : \pi_1 = \pi_2 = \ldots = \pi_5 = 0.2$, corresponding to Test 1, was performed with a resulting value of $\chi^2 = 1.4315 \times 10^4$ with four degrees of freedom, a highly significant value. Also, a test of $H_{02} : \theta = 2.5$, corresponding to Test 2, was performed, resulting in the highly significant value of $\chi^2 = 2.3041 \times 10^2$ based on one degree of freedom.

A large sample approximation to the dispersion matrix of $(\hat{\theta}, p_1, p_2, \ldots, p_5)$ was calculated by replacing the parameters $\theta, \pi_1, \pi_2, \ldots, \pi_5$ with their maximum likelihood estimates in $(1/50)\Sigma$, where Σ is defined in (5.1.16). These variances and covariances can be used for calculating approximate asymptotic confidence bounds for the parameters of the model.

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BIOGRAPHICAL SKETCH

Robert John Beaver was born March 27, 1937, at Mount Carmel, Pennsylvania, moving at an early age to Kulpmont, a small coal-mining town in the anthracite region of Pennsylvania. In June, 1955, he was graduated from Kulpmont High School. In June, 1959, he received the degree of Bachelor of Science in Secondary Education from Bloomsburg State College. He taught in the public schools of Pennsylvania until June, 1963, when he enrolled in the Graduate School of Bucknell University, Lewisburg, Pennsylvania, where he was awarded the degree of Master of Science with a major in Mathematics in June, 1964. He then enrolled in the Graduate School of the University of Florida, being awarded the degree of Master of Statistics in June, 1966. After working as an instructor in the Department of Statistics at the University of Florida until June, 1968, he returned to graduate study and,until the present time,has pursued his work toward the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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